Math 54 - Homework 8 Solutions

5.4.2

$$ker(\mathbf{A^T}) = ker\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\right)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 0$$

$$\text{Basis for } ker(\mathbf{A}^T) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

For the sketch illustrating $(Im(\mathbf{A}))^\perp = ker(\mathbf{A}^T)$, $Im(\mathbf{A})$ is the plane spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Notice that $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

5.4.7

$$\mathbf{A} \text{ is a } n \times n \text{ symmetric matrix. This implies that } \mathbf{A} = \mathbf{A}^T$$

$$(Im(\mathbf{A}))^\perp = ker(\mathbf{A}^T) = ker(\mathbf{A})$$

$$\text{Or alternatively, } Im(\mathbf{A}) = (ker(\mathbf{A}))^\perp$$

5.4.8

$$\mathbf{A} \text{ has dimension } m \times n$$

$$\mathbf{y} \text{ has dimension } m \times 1$$

$$ker(\mathbf{A}) = 0 \Rightarrow \mathbf{A}^T\mathbf{A} \text{ is invertible}$$

$$\text{Least squares solution of } L(\mathbf{x}) = \mathbf{y} \text{ is } (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y}$$

(a)

$$L^+(\mathbf{y}) = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y}$$

$$\mathbf{A}^+ = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$$

$$\text{It is easy to verify that } L^+ \text{ is linear}$$

$$L^+(\mathbf{y} + \mathbf{z}) = L^+(\mathbf{y}) + L^+(\mathbf{z})$$

$$L^+(k\mathbf{y}) = kL^+(\mathbf{y})$$

(b)

$$\text{If A is invertible, then } (\mathbf{A}^T\mathbf{A})^{-1} = A^{-1}(A^T)^{-1}$$

$$\text{then, } \mathbf{A}^+ = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$$

$$= \mathbf{A}^{-1}(\mathbf{A}^T)^{-1}\mathbf{A}^T$$

$$= \mathbf{A}^{-1}$$

$$\text{If } L \text{ is invertible, } L^+ = L^{-1}$$

(c)

$$L^+(L(x)$$

$$L^+(\mathbf{A}x)$$

$$(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{A}x$$

$$= x$$

(d)

$$L(L^+(\mathbf{y})$$

$$= L((\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y})$$

$$= \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y}$$

5.4.10 (a)

Let $\mathbf{x}$ be a solution to the system $\mathbf{x} = x_h + x_0$ where $x_h \in ker(\mathbf{A}), x_0 \in (ker(\mathbf{A}))^\perp$

$$\mathbf{A}\mathbf{x} = b$$

$$\mathbf{A}(x_h + x_0) = b$$

$$\mathbf{A}x_h + \mathbf{A}x_0 = b$$

$$\mathbf{A}x_0 = b$$

(b)

$$\mathbf{A}x_0 = b$$

$$\mathbf{A}x_1 = b$$

where $x_0, x_1 \in (ker(\mathbf{A}))^\perp$

$$\mathbf{A}(x_0 - x_1)$$

$$= \mathbf{A}x_0 - \mathbf{A}x_1$$

$$= b - b = 0$$

which is a contradiction. It would imply that $x_0 - x_1 \in ker(\mathbf{A})$

(c)

If x_1 is a solution, it can be written as $x_1 = \mathbf{x}_{ker(\mathbf{A})} + \mathbf{x}_{ker(\mathbf{A})^\perp}$

However, in (b) we established that $\mathbf{x}_{ker(\mathbf{A})^\perp}$ has only one possible value, x_0

Thus, $x_1 = \mathbf{x}_{ker(\mathbf{A})} + x_0$

Look at $\|x_1\|^2 = (\mathbf{x}_{ker(\mathbf{A})} + x_0) \cdot (\mathbf{x}_{ker(\mathbf{A})} + x_0)$

$= x_0 \cdot x_0 +$ something positive $= \|x_0\|^2 +$ something positive

It follows that $\|x_1\|^2 > \|x_0\|^2$

5.4.15

$$\mathbf{A} \text{ is an } m \times n \text{ matrix}$$

$$ker(\mathbf{A}) = \{\mathbf{0}\}$$

$$\mathbf{A}^T\mathbf{A} \text{ is invertible by fact } \textit{5.4.2b}$$

$$\mathbf{B} \cdot \mathbf{A} = \mathbf{I}$$

$$\mathbf{A}^T \cdot \mathbf{B}^T = \mathbf{I}$$

$$\mathbf{A}^T\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1} = \mathbf{I}$$

$$\text{Let } \mathbf{B} = (\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1})^T = (\mathbf{A}^T\mathbf{A})^{-1}A^T$$

5.4.22

$$x^* = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T b$$

$$= \left(\begin{bmatrix} 3 & 5 & 4 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 3 & 5 & 4 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$b - \mathbf{A}x^* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

5.5.2

$$< f, g+h >=< g+h, f >=< g, f > + < h, f >=< f, g > + < f, h >$$

5.5.4 (a)

$$< A, B >= tr(A^T B)$$

$$= tr\left([a_1 \ldots a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right)$$

$$= tr([a_1 \cdot b_1 + \ldots + a_n \cdot b_n])$$

You can think of this as a one by one matrix

The trace of a square matrix is the sum of its diagonal matrix.

Thus, $< A, B >$ is the dot product of A and B

(b)

$$< A, B >= tr\left(\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} [b_1 \ldots b_m] \right)$$

$$= a_1 \cdot b_1 + \ldots a_m \cdot b_m$$

5.5.10

$$< f, g >= \frac{1}{2} \int_{-1}^{1} f(t) g(t)\, dt$$

$$< f, g(t) >= \frac{1}{2} \int_{-1}^{1} t(at^2 + bt + c)\, dt$$

$$= \frac{1}{2} \left[\frac{9t^4}{4} + \frac{bt^3}{3} + \frac{ct^2}{2} \right]_{-1}^{1}$$

$$\text{set } < t, g(t) >= 0, \text{ we get } \frac{2b}{3} = 0 \text{ or } b = 0$$

$$g(t) = at^2 + c$$

A basis for the space of all functions in P_2 that are orthogonal to $f(t) = t$ is $\{1, t^2\}$.

Apply Gram-Schmidt to the basis $\{v_1, v_2\}$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{\frac{1}{2}\int_{-1}^{1} dt}} = 1$$

$$u_2 = \frac{v_2^{\perp}}{\|v_2^{\perp}\|} \text{ where } v_2^{\perp} = v_2 - (u_1 \cdot v_2)u_1 = t^2 - \frac{1}{3}$$

$$\|v_2^{\perp}\|^2 =< t^2 - \frac{1}{3}, t^2 - \frac{1}{3} >$$

$$=< t^2, t^2 > +2 < t^2, -\frac{1}{3} > + < -\frac{1}{3}, -\frac{1}{3} >$$

$$= \frac{1}{5} - \frac{2}{9} + \frac{1}{9} = \frac{1}{5} - \frac{1}{9} = \frac{9-5}{45} = \frac{4}{45}$$

$$u_2 = \frac{45}{4}(t^2 - \frac{1}{3})$$

5.5.12

$$f(t) = |t|$$

$$b_k =< f(t), \sin(kt) >$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |t| \sin(kt)\, dt$$

$$c_k =< f(t), \cos(kt) >= \frac{1}{\pi}\int_{-\pi}^{\pi} |t| \cos(kt)\, dt$$

$$a_0 =< f(t), \frac{1}{\sqrt{2}} >= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} |t|\, dt = \frac{1}{\sqrt{2\pi}}\pi^2$$

5.5.16

$$P_1 \text{ with inner product } < f, g >= \int_0^1 f(t)g(t)\, dt$$

(a)

$$\text{Basis for } P_1 = \{v_1, v_2\} = \{1, t\}$$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{\int_0^1 dt}} = 1$$

$$u_2 = \frac{v_2^{\perp}}{\|v_2^{\perp}\|} = \sqrt{12}(t - \frac{1}{2})$$

$$\text{where } v_2^{\perp} = v_2 - (u_1 \cdot v_2)u_1 = (t - \frac{1}{2})$$

$$\|v_2^{\perp}\| = \sqrt{\int_0^1 t^2 - t + \frac{1}{4}\,dt}$$
$$= \sqrt{\left[\frac{t^3}{3} - \frac{t^2}{2} + \frac{1}{4}t\right]_0^1}$$
$$= \sqrt{\frac{1}{3} - \frac{1}{2} + \frac{1}{4}}$$
$$= \sqrt{\frac{1}{12}}$$

(b)

$$\text{We need to find } proj_{p_1}(f(t))$$

$$proj_{p_1}(f) = < u_1, f > u_1 + < u_2, f > u_2$$
$$= < 1, t^2 > 1 + \left\langle \sqrt{12}\left(t - \frac{1}{2}\right), t^2 \right\rangle \sqrt{12}(t - \frac{1}{2})$$

5.2
This is **FALSE**. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \neq \mathbf{A}^T\mathbf{B}^T$ unless $\mathbf{B}^T$ and $\mathbf{A}^T$ commute.

5.4
This is **TRUE**.

$$\mathbf{A} \text{ and } \mathbf{S} \text{ are orthongonal}$$

$$\|\mathbf{S}^{-1}\mathbf{AS}x\| = \|\mathbf{SS}^{-1}\mathbf{AS}x\|$$
$$= \|\mathbf{AS}x\|$$
$$= \|\mathbf{S}x\|$$
$$= \|x\|$$

5.10
This is **TRUE**.

$$\mathbf{AA}^{-1} = \mathbf{I}$$
$$(\mathbf{AA^{-1}})^T = \mathbf{I}^T$$
$$(\mathbf{A}^{-1})^T\mathbf{A}^T = \mathbf{I}$$

5.20
$ker(\mathbf{A}) = 0$ only guarantees that the column vectors are linearly independent. However, by fact *5.3.10*, we also necessitate that the column vectors have length 1 and are

orthogonal to each other.

5.22 This is **TRUE** because $\Re^n$ is an inner product space, and also by the Gram-Schmidt process.

6.1.2

$$det \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = 2 * 5 - 3 * 4 = -2$$

6.1.6

$$det \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix} = 6 * 4 * 1 = 24$$

6.1.12

$$det \begin{bmatrix} 1 & k \\ k & 4 \end{bmatrix} = 4 - k^2$$

This is invertible only if $k \neq \pm 2$

6.1.22

$$det \begin{bmatrix} \cos(k) & 1 & -\sin(k) \\ 0 & 2 & 0 \\ \sin(k) & 0 & \cos(k) \end{bmatrix}$$

$$= \cos(k)(2\cos(k)) - 1 \times 0 - \sin(k)(-2\sin(k))$$

$$= 2(\cos^2(k) + sin^2(k)) = 2$$

6.1.40

There are lots of zeroes everywhere. This is good.

$$det = 3 * -2 * -4 * -5$$

$$= -120$$

6.1.44

For n by n matrix A: $det(k\mathbf{A}) = k^n * det(\mathbf{A})$.

Proof by induction works here. When n=1 or 2, this is easy to see. Assume true for n=m-1, show statement is true for n=m.

6.1.55 (a)

$$\text{Notice that } d_4 = d_3 - det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= d_3 - d_2$$

$$\text{In general, } d_n = d_{n-1} - d_{n-2}$$

(b)

$$d_1 = 1; d_2 = 0$$

The rest follows from the recursive formula.

$$\underbrace{1, 0, -1, -1, 0, 1}, \underbrace{1, 0, -1, -1, 0, 1}, \ldots$$

Note that the the sequence repeats

It fllows from part (a) that 100 mod 6 = 4 so, $d_{100} = d_4 = -1$