Solutions: Assignment 4

3.3.20 *Find the redundant column vectors of the given matrix A "by inspection". Then find a basis of the image of A and a basis of the kernel of A.*

$$A = \begin{bmatrix} 1 & 0 & 5 & 3 & -3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The second and third columns are mutliples of the first. And the fifth column is 3 times the third column minus 12 times the first. So the second, third, and fifth columns are redundant. And a basis for the image is just $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. To get a basis for the kernel we look at $A\overline{x} = \overline{0}$. This tells us that relationship between entries of $\overline{x}$ are just $x_1 = -5x_3 - 3x_4 + 3x_5$ and $x_4 = -3x_5$. So this gives us a basis of 3 elements:

$$\begin{aligned} x_2 &= 1 \\ x_3 &= 0 \\ x_5 &= 0 \end{aligned} \qquad \begin{aligned} x_2 &= 0 \\ x_3 &= 1 \\ x_5 &= 0 \end{aligned} \qquad \begin{aligned} x_2 &= 0 \\ x_3 &= 0 \\ x_5 &= 1 \end{aligned}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

3.3.22 *Find the reduced row-echelon form of the given matrix A. Then find a basis of the image of A and a basis for the kernel of A.*

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 8 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -15 \\ 0 & -5 & -25 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & -5 & -25 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

There are leading ones in the first two columns of $rref(A)$, So a basis for the image is $\begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}$. We know that $\overline{x}$ is in the kernel is equivalent to $A\overline{x} = \overline{0}$. That's equivalent to $rref(A)\overline{x} = \overline{0}$. Which is equivalent to $x_1 = 6x_3$ and $x_2 = -5x_3$. Which is equivalent to $\overline{x} = x_3 \begin{bmatrix} 6 \\ -5 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} 6 \\ -5 \\ 1 \end{bmatrix}$ is a basis for the kernel.

3.3.33 *A subspace V of $\mathbb{R}^n$ is called a hyperplane if V is defined by the homogeneous equation*

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n = 0$$

where at least one of the coefficients c_i is nonzero. What is the dimension of a hyperplane in $\mathbb{R}^n$? Justify your answer carefully. What is a hyperplane in $\mathbb{R}^3$? What is it in $\mathbb{R}^2$.

Define $C = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix}$. Notice that $\overline{v}$ is in V exactly when $C\overline{v} = \overline{0}$. So $ker(C) = V$. Call the first nonzero entry of C by k. Then $rref(C) = \frac{1}{k}C$. So $rref(C)$ has one leading nonzero, i.e. $rank(C) = 1$. By the Rank-Nullity Theorem, $dim(ker(C)) + rk(C) = n$. So $dim(V) = dim(ker(C) = n - rk(C) = n - 1$. So in $\mathbb{R}^3$, a hyperplane is $3 - 1 = 2$ dimensional, or a plane. In $\mathbb{R}^2$, a hyperplane is $2 - 1 = 1$ dimensional – it's a line.

3.3.39 *We are told that a certain 5×5 matrix A can be written as*

$$A = BC$$

where B is 5×4 and C is 4×5. Explain how you know that A is not invertible.

Since C is 4×5, we know that $im(C)$ is a subspace of $\mathbb{R}^4$. So $dim(im(C)) \leq 4$. By the Rank-Nullity theorem, we know $dim(ker(C)) + dim(Im(C)) = 5$. This means $dim(ker(C)) \geq 1$. So there is some vector $\overline{v} \neq \overline{0}$ in the kernel of C.

Now for any 5×5 matrix E at all, $EA\overline{v} = EBC\overline{v} = EB\overline{0} = \overline{0} \neq \overline{v}$. So no E can make it true that $EA = I_5$. In other words, A is not invertible.

3.3.52 *Find a basis for the row space of the matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \end{bmatrix}$$

Row reducing does not change the row space. So we want to row reduce A:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The two remaining nonzero rows are not multiples of each other. So a basis for the row space is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$.

3.3.56 *An $n \times n$ matrix A is called nilpotent if $A^m = 0$ for some positive integer m. Examples are triangular matrices whose entries on the diagonal are all 0. Consider a nilpotent $n \times n$ matrix A and choose the small number m such that $A^m = 0$. Pick a vector $\overline{v}$ in $\mathbb{R}^n$ such that $A^{m-1}\overline{v} \neq \overline{0}$. Show that the vectors $\overline{v}, A\overline{v}, A^2\overline{v}, \ldots, A^{m-1}\overline{v}$ are linearly independent.*

Suppose that

$$\overline{0} = c_0\overline{v} + c_1 A\overline{v} + c_2 A^2\overline{v} + \ldots + c_{m-1}A^{m-1}\overline{v}$$

If all the c's before c_i were 0, we will show that $c_i = 0$ also. This means that all the c's must be 0.

Consider the situation where all the c's before c_i are 0. Then we get:

$$\overline{0} = c_i A^i\overline{v} + c_{i+1}A^{i+1}\overline{v} + \ldots + c_{m-1}A^{m-1}\overline{v}$$

If we multiply this equation by A^{m-i-1} we get:

$$\begin{aligned} \overline{0} &= c_i A^{m-1}\overline{v} + c_{i+1}A^m\overline{v} + \ldots + c_{m-1}A^{2m-i-2}\overline{v} \\ &= c_i A^{m-1}\overline{v} + A^m\left(c_{i+1}\overline{v} + c_{i+2}A\overline{v} + \ldots + c_{m-1}A^{m-i-2}\overline{v}\right) \\ &= c_i A^{m-1}\overline{v} + 0\left(c_{i+1}\overline{v} + c_{i+2}A\overline{v} + \ldots + c_{m-1}A^{m-i-2}\overline{v}\right) \\ &= c_i A^{m-1}\overline{v} \end{aligned}$$

But we picked $\overline{v}$ so that $A^{m-1}\overline{v} \neq \overline{0}$. Thus c_i must also be 0, which results in all the c's being 0. So the vectors $\overline{v}, A\overline{v}, A^2\overline{v}, \ldots, A^{m-1}\overline{v}$ are linearly independent.

3.4.20 *Find the matrix of the linear transformation $T(\overline{x}) = A\overline{x}$ with respect to the basis $\mathfrak{B} = (\overline{v}_1, \overline{v}_2)$.*

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \overline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \overline{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{aligned} B &= S^{-1}AS = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \tfrac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

3.4.26 *Find the matrix of the linear transformation $T(\overline{x}) = A\overline{x}$ with respect to the basis $\mathfrak{B} = (\overline{v}_1, \ldots, \overline{v}_m)$.*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}; \overline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \overline{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} B &= S^{-1}AS = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 4 & 6 \end{bmatrix} \end{aligned}$$

3.4.38 *Find a basis $\mathfrak{B}$ of $\mathbb{R}^n$ such that the $\mathfrak{B}$-matrix B of the given linear transformation T is diagonal.*

T is the reflection about the line in $\mathbb{R}^2$ spanned by $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

We want to pick vectors $\overline{v}$ so that $T(\overline{v}) = c_{\overline{v}}\overline{v}$ for some $c_{\overline{v}}$. that way, the off-diagonal entries of B will be zero. Vectors on the line don't move, so $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = 1\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Also, if we pick something perpendicular to the line, its reflection is just its negation. So $T\left(\begin{bmatrix} -3 \\ 2 \end{bmatrix}\right) = (-1)\begin{bmatrix} -3 \\ 2 \end{bmatrix}$. So according to the basis

$$\mathfrak{B} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

we will get

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which is of the form we want.

Ch.3 Ex. *True or False?*

2 *The span of vectors $\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n$ consists of all linear combinations of vectors $\overline{v}_1, \overline{v}_2, \ldots$.*
True. This is the definition of span.

7 *If $2\overline{u} + 3\overline{v} + 4\overline{w} = 5\overline{u} + 6\overline{v} + 7\overline{w}$, then the vectors $\overline{u}$, $\overline{v}$, $\overline{w}$ must be linearly dependent.*
True. $\overline{w} = -\overline{u} - \overline{v}$, so $\overline{w}$ is redundant.

10 *If A is a 5×6 matrix of rank 4, then the nullity of A is 1.*
False. We should get rank plus nullity equalling 6, not 5.

18 *The vectors of the form $\begin{bmatrix} a \\ b \\ 0 \\ a \end{bmatrix}$ (where a and b are arbitrary real numbers) form a subspace of $\mathbb{R}^4$.*
True. This set includes $\overline{0}$ and is closed under both addition and multiplication.

4.1.2 *Is the following subset of P_2 a subsapce? Find a basis for it if it is.*

$$\{p(t)|\, p(2) = 0\}$$

Yes. $0(2) = 0$ so 0 is included. If $p_1(2) = 0$ and $p_2(2) = 0$ then $(p_1 + p_2)(2) = p_1(2) + p_2(2) = 0 + 0 = 0$ so it is closed under addition. And if $p(2) = 0$ then $(rp)(2) = rp(2) = r0 = 0$, so it is closed under scalar multiplication. Since $dim\,(P_2) = 3$ and 1 is not in the subspace, we know the dimension of the subspace must be at most 2. Since $x - 2$ and $x^2 - 4$ are in the subspace and linearly independent, they must be a basis.

4.1.6 *Is this subset of $\mathbb{R}^{3\times3}$ a subspace?*

The invertible 3×3 matrices No. This set actually fails all three requirements of a subspace. It does not have 0. It has both I_3 and $-I_3$, but it doesn't $I_3 + (-I_3) = 0$, so it isn't closed under addition. It has I_3 but it does not have $0 \cdot I_3 = 0$, so it is not closed under multiplication either.

4.1.20 *Find a basis for the space of all matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathbb{R}^{2\times2}$ such that $a = d$.*

If A is as written above, then

$$A = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

As those three matrices are linearly independent, we get a basis of

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

So the dimension is 3.

4.1.34 *If B is a diagonal 4×4 matrix, what are the possible dimensions of the space of all 4×4 matrices A that commute with B.*

$$B = \begin{bmatrix} b_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & b_4 \end{bmatrix}$$

We need to have that $(AB)_{ij} = (BA)_{ij}$. This tells us that $A_{ij}b_j = b_iA_{ij}$. This tells us $(b_j - b_i)A_{ij} = 0$. I.e. $b_i = b_j$ or $A_{ij} = 0$. So we can make a basis out of matrices E_{ij} which have a 1 in the ijth position and 0 elsewhere, only including the E_{ij}s for which $b_i = b_j$. If all the b's are the same, we include all 16 E's. If 3 of the b's are the same, we include the 9 E's for those 3 along with the E for the remaining B with itself. This totals to 10. If there are two pairs of b's, then inside for each pair we have 4 E's. If there is one pair, and two other values, we get 4 E's for the pair plus one for each of the others, totalling 6. And finally, if all for b's are different, we get one E value for each, for 4 total. So the possible dimensions of this space are 4, 6, 10 or 16.