SOLUTIONS: ASSIGNMENT 3

2.4.15 *Compute the matrix product*

$$\begin{bmatrix} 1 & -2 & -5 \\ -2 & 5 & 11 \end{bmatrix} \begin{bmatrix} 8 & -1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}.$$

Explain why the result does not contradict Fact 2.4.9.

$$\begin{bmatrix} 1 & -2 & -5 \\ -2 & 5 & 11 \end{bmatrix} \begin{bmatrix} 8 & -1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 8-2-5 & -1-4+5 \\ -16+5+11 & 2+10-11 \end{bmatrix} = I_2.$$

Fact 2.4.9 only applies if the two matrices being multiplied are $n \times n$. Since these two matrices are 2×3 and 3×2 they need not be invertible.

2.4.20 *True / False: For two invertible $n\times n$ matrices A and B, $(A-B)(A+B) = A^2 - B^2$.*

False.

$$(A-B)(A+B) = A^2 - B^2 + AB - BA \neq A^2 - B^2$$

because matrix multiplication is noncommutative.

2.4.23 *True / False: $(ABA^{-1})^3 = AB^3A^{-1}$.*

True.

$$(ABA^{-1})^3 = (ABA^{-1})(ABA^{-1})(ABA^{-1}) = AB^3A^{-1}$$

since matrix multiplication is associative.

2.4.26 *Use the given partition to compute the product. Check your work by computing the same product without using a partition.*

$$\begin{aligned} AB &= \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 2 & 2 & 3 \\ 3 & 4 & 4 & 5 \\ \hline 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{array}\right] \\ &= \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right] \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline 0 & B_{22} \end{array}\right] \\ &= \left[\begin{array}{c|c} A_{11}B_{11} & A_{11}B_{12} + A_{12}B_{22} \\ \hline 0 & A_{22}B_{22} \end{array}\right] \end{aligned}$$

Using the partition,

$$A_{11}B_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A_{11}B_{12} + A_{12}B_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}$$

$$A_{22}B_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

So

$$AB = \left[\begin{array}{cc|cc} 1 & 2 & 3 & 5 \\ 3 & 4 & 7 & 9 \\ \hline 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{array}\right].$$

Without using the partition,

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 & 3 \\ 3 & 4 & 4 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 2+1 & 3+2 \\ 3 & 4 & 4+3 & 5+4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 3 & 5 \\ 3 & 4 & 7 & 9 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix} \end{aligned}$$

2.4.41 *Consider the matrix*

$$D_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

We know that the linear transformation $T(\overrightarrow{x}) = D_\alpha \overrightarrow{x}$ *is a counterclockwise rotation through an angle* α.

(a) *For two angles,* α *and* β*, consider the products* $D_\alpha D_\beta$ *and* $D_\beta D_\alpha$. *Arguing geometrically, describe the linear transformations* $\overrightarrow{y} = D_\alpha D_\beta \overrightarrow{x}$ *and* $\overrightarrow{y} = D_\beta D_\alpha \overrightarrow{x}$. *Are the two transformations the same?*

The transformation $\overrightarrow{y} = D_\alpha D_\beta \overrightarrow{x}$ represents a counterclockwise rotation by the angle β, followed by a counterclockwise rotation by the angle α. On the other hand, the transformation $\overrightarrow{y} =$

$D_\beta D_\alpha \vec{x}$ represents a counterclockwise rotation by the angle α followed by a counterclockwise rotation by the angle β. Both transformations are a counterclockwise rotation by $\alpha+\beta$ so they are the same.

(b) *Now compute the products $D_\alpha D_\beta$ and $D_\beta D_\alpha$. Do the results make sense in terms of your answer in part (a)? Recall the trigonometric identities*

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\begin{aligned}
D_\alpha D_\beta &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \\
&= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix} \\
&= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} \\
D_\beta D_\alpha &= \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \\
&= \begin{bmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha & -\cos\beta\sin\alpha - \sin\beta\cos\alpha \\ \sin\beta\cos\alpha + \cos\beta\sin\alpha & -\sin\beta\sin\alpha + \cos\beta\cos\alpha \end{bmatrix} \\
&= \begin{bmatrix} \cos(\beta+\alpha) & -\sin(\beta+\alpha) \\ \sin(\beta+\alpha) & \cos(\beta+\alpha) \end{bmatrix}
\end{aligned}$$

So $D_\alpha D_\beta = D_\beta D_\alpha$.

2.8 True / False: The function $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 1 \end{bmatrix}$ is a linear transformation.

False, T is not a linear transformation because it does not satisfy $T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

2.15 *True/False: The matrix $\begin{bmatrix} k & -2 \\ 5 & k-6 \end{bmatrix}$ is invertible for all real numbers k.*

True. A 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if its determinant $ad - bc \neq 0$. The determinant of this matrix is

$k(k-6) + 10 = k^2 - 6k + 10 = (k^2 - 6k + 9) + 1 = (k-3)^2 + 1 \neq 0.$

2.40 *True/False: If A^2 is invertible, then the matrix A itself is invertible.*

True. If A^2 is invertible, there is a matrix B such that $(A^2)B = I$. Since matrix multiplication is associative, $A(AB) = I$. In other words, A is invertible and AB is its inverse.

2.46 *If A is an $n \times n$ matrix such that $A^2 = 0$, then the matrix $I_n + A$ must be invertible.*

True, since $(I_n + A)(I_n - A) = I_n + A - A - A^2 = I_n$, so $I_n - A$ is the inverse of $I_n + A$.

3.1.8 *Find vectors that span the kernel of the matrix* $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$.

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}.$$

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

$$\begin{aligned} x_3 &= t \\ x_1 = x_3 &= t \\ x_2 = -2x_3 &= -2t \end{aligned}$$

The kernel of A is $\begin{bmatrix} t \\ -2t \\ t \end{bmatrix}$ so it is spanned by $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

3.1.23 *Describe the image and kernel of this transformation geometrically: reflection about the line $y = \frac{x}{3}$ in $\mathbb{R}^2$.*

Reflection is its own inverse so this transformation is invertible. Its image is $\mathbb{R}^2$ and its kernel is $\{\vec{0}\}$.

3.1.32 *Give an example of a linear transformation whose image is the line spanned by*

$$\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$$

in $\mathbb{R}^3$.

The image of a transformation $T(\overrightarrow{x}) = A\overrightarrow{x}$ is the span of the column vectors of A, we can simply take

$$A = \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}.$$

3.1.42 *Express the image of the matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 2 \\ 1 & 4 & 7 & 0 \end{bmatrix}$$

as the kernel of a matrix B.

Following the hint, the image of A is the set of vectors $\overrightarrow{y}$ in $\mathbb{R}^4$ such that the system $A\overrightarrow{x} = \overrightarrow{y}$ is consistent. Writing this system explicitly in terms of coordinates,

$$\left|\begin{array}{ccccccccccc} x_1 & + & x_2 & + & x_3 & + & 6x_4 & = & y_1 & & & \\ x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & = & & y_2 & & \\ x_1 & + & 3x_2 & + & 5x_3 & + & 2x_4 & = & & & y_3 & \\ x_1 & + & 4x_2 & + & 7x_3 & & & = & & & & y_4 \end{array}\right|$$

Reducing to row-echelon form,

$$\left|\begin{array}{cccccccccccccc} x_1 & & & - & x_3 & + & 8x_4 & = & & & & 4y_3 & - & 3y_4 \\ & & x_2 & + & 2x_3 & - & 2x_4 & = & & & - & y_3 & + & y_4 \\ & & 0 & & & & & = & y_1 & & - & 3y_3 & + & 2y_4 \\ & & 0 & & & & & = & & y_2 & - & 2y_3 & + & y_4 \end{array}\right|$$

If $\overrightarrow{y}$ is a vector in $\mathbb{R}^4$, we can always choose the appropriate $\overrightarrow{x}$ so that the first two equations are true, so the system is consistent if and only if $\overrightarrow{y}$ is a solution to the last two equations. In other words, $\overrightarrow{y}$ is in the kernel of the matrix

$$B = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix}.$$

3.1.52 *Consider a $p \times m$ matrix A and a $q \times m$ matrix B, and form the partitioned matrix*

$$C = \begin{bmatrix} A \\ B \end{bmatrix}.$$

What is the relationship between $\ker(A)$, $\ker(B)$, *and* $\ker(C)$*?*

If $\overrightarrow{x}$ is a vector in $\mathbb{R}^m$, then $C\overrightarrow{x} = 0$ if and only if both $A\overrightarrow{x} = 0$ and $B\overrightarrow{x} = 0$. Therefore

$$\ker(C) = \ker(A) \cap \ker(B).$$

3.2.2 *Is W a subspace of $\mathbb{R}^3$?*

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \le y \le z \right\}$$

No, because W is not closed under scalar multiplication. For example, $(0,0,1)$ is in W, but $-1(0,0,1) = (0,0,-1)$ is not in W.

3.2.5 *Give a geometrical description of all subspaces of $\mathbb{R}^3$. Justify your answer.*

A subspace of $\mathbb{R}^3$ is either $\mathbb{R}^3$ itself, a plane containing the origin, a line through the origin, and the origin itself. This is because a basis of $\mathbb{R}^3$ can contain either $3, 2, 1$ or 0 linearly independent vectors, respectively.

3.2.34 *Consider the 5×4 matrix*

$$A = \begin{bmatrix} | & | & | & | \\ \overrightarrow{v_1} & \overrightarrow{v_2} & \overrightarrow{v_3} & \overrightarrow{v_4} \\ | & | & | & | \end{bmatrix}.$$

We are told that the vector $\overrightarrow{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is in the kernel of A. Write $\overrightarrow{v_4}$ as a linear combination of $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$.

$$\begin{aligned} 0 = A\overrightarrow{x} &= \overrightarrow{v_1} + 2\overrightarrow{v_2} + 3\overrightarrow{v_3} + 4\overrightarrow{v_4} \\ \overrightarrow{v_4} &= -\frac{1}{4}\overrightarrow{v_1} - \frac{2}{4}\overrightarrow{v_2} - \frac{3}{4}\overrightarrow{v_3} \end{aligned}$$

3.2.36 *Consider a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^p$ and some linearly dependent vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \cdots, \overrightarrow{v_m}$ in $\mathbb{R}^n$. Are the vectors $T(\overrightarrow{v_1}), T(\overrightarrow{v_2}), \cdots, T(\overrightarrow{v_m})$ necessarily linearly dependent? How can you tell?*

The $T(\overrightarrow{v_i})$ are linearly dependent because they satisfy the same linear dependence relation as the $\overrightarrow{v_i}$. Suppose that $a_1, a_2, \cdots, a_m$ are constants, not all 0, such that

$$a_1\overrightarrow{v_1} + a_2\overrightarrow{v_2} + \cdots + a_m\overrightarrow{v_m} = 0.$$

Then

$$\begin{aligned} 0 &= T(a_1\overrightarrow{v_1} + a_2\overrightarrow{v_2} + \cdots + a_m\overrightarrow{v_m}) \\ &= T(a_1\overrightarrow{v_1}) + T(a_2\overrightarrow{v_2}) + \cdots + T(a_m\overrightarrow{v_m}) \\ &= a_1T(\overrightarrow{v_1}) + a_2T(\overrightarrow{v_2}) + \cdots + a_mT(\overrightarrow{v_m}) \end{aligned}$$