Math 54 - Homework 14 Solutions

9.6.2

$$\mathbf{A} = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}$$

$$\lambda_1 = a + bi \text{ and } \lambda_2 = a - bi$$

$$\lambda_1 + \lambda_2 = 0 \Rightarrow 2a = 0$$

$$\lambda_1 \lambda_2 = 1 \Rightarrow a^2 + b^2 = 1$$

$$\lambda_1 = i,\ \lambda_2 = -i$$

$$(\mathbf{A} - \lambda_1 \mathbf{I}) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives}$$

$$\begin{bmatrix} -2-i & -5 \\ 1 & 2-i \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$z_1 = -5s, z_2 = (2+i)s,\ z = \begin{bmatrix} -5 \\ 2+i \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\mathbf{A}$ has complex conjugate eigenvalues $\pm i$ with corresponding eigenvectors $\begin{bmatrix} -5 \\ 2 \end{bmatrix} \pm i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

General Solution:

$$c_1 \left(\cos(t) \begin{bmatrix} -5 \\ 2 \end{bmatrix} - \sin(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + c_2 \left(\sin(t) \begin{bmatrix} -5 \\ 2 \end{bmatrix} - \cos(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

9.6.4

$$\mathbf{A} = \begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & 3 \end{bmatrix}$$

Eigenvalues are $2, 2+i, 2-i$

The corresponding eigenvectors are $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2-i \\ 5 \end{bmatrix} \begin{bmatrix} 5 \\ -2+i \\ 5 \end{bmatrix}$

$$\text{The general solution is } c_1 e^{2t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_2 \left(e^{2t}\cos(t) \begin{bmatrix} 5 \\ -2 \\ 5 \end{bmatrix} - e^{2t}\sin(t) \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) +$$
$$c_3 \left(e^{2t}\sin(t) \begin{bmatrix} 5 \\ -2 \\ 5 \end{bmatrix} + e^{2t}\cos(t) \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right)$$

9.6.8

The eigenvalues are: $-1, 1, 2+3i, 2-3i$

The corresponding eigenvectors are: $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2+3i \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2-3i \end{bmatrix}$

$$\text{The general solution is } c_1 e^{-t} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \left(e^{2t}\cos(3t) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} - e^{2t}\sin(3t) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix} \right)$$
$$+ c_4 \left(e^{2t}\sin(3t) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} - e^{2t}\cos(3t) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix} \right)$$

9.6.14

$$\text{The eigenvalues are: } 2, 1+i, 1-i$$

$$\text{The corresponding eigenvectors are: } + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$x(t) = c_1 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \left(e^t \cos(t) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - e^t \sin(t) \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right) + c_3 \left(e^t \sin(t) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e^t \cos(t) \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right)$$

(a)

$$x(0) = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$x(0) = c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$c_1 = 2, c_2 = -2, c_3 = -1$$

$$\text{Plug these values back into } x(t)$$

9.6.16

Follow the hint: plug in the equations (2) and (3) for $w_1(t)$ and $w_2(t)$. If the equations hold, then $x_1(t)$ and $x_2(t)$ consist of linear combinations and $w_1(t)$ and $w_2(t)$.

10.2.2

$$r^2 - 6r + 5 = 0$$

$$(r-1)(r-5) = 0$$

$$y(t) = c_1 e^t + c_2 e^{5t}$$

$$y(0) = c_1 + c_2 = 1$$

$$y(2) = c_1 e^2 + c_2 e^{10} = 1$$

$$\begin{bmatrix} 1 & 1 \\ e^2 & e^{10} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Now solve for c_1 and c_2. We are guaranteed unique solutions because the determinant is not zero, implying that the matrix is invertible

10.2.6

$$y'' + y = 0, r = \pm i$$

$$r^2 + 1 = 0, y = c_1 \cos(t) + t_2 \sin(t)$$

$$y(0) = c_1 = 0$$

$$y(2\pi) = c_1 = 1$$

There is no solution

10.2.8

$$r^2 - 2r + 1 = 0, y = c_1 e^t + c_2 t e^t$$

$$(r-1)(r-1) = 0$$

$$y(-1) = c_1 e^{-1} - c_2 e^{-1} = (c_1 - c_2)e^{-1} = 0 \Rightarrow c_1 - c_2 = 0$$

$$y(1) = (c_1 + c_2)e = 2$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{2}{e} \end{bmatrix}$$

Now solve for c_1, c_2

10.2.14

$$r^2 - 2r + \lambda = 0$$

$$\text{The quadratic equation gives: } \frac{2 \pm \sqrt{4 - 4\lambda}}{2} = 1 \pm \sqrt{1 - \lambda}$$

When $\lambda = 1,$ or $\lambda < 1,$ the solutions are trivial. Consider $\lambda > 1$

$$1 \pm i\sqrt{\lambda - 1}$$

$$y(t) = c_1 e^t \cos(\sqrt{\lambda - 1} t) + c_2 e^t \sin(\lambda - 1t)$$

$$y(0) = c_1 = 0$$

$$y(\pi) = c_2 e^{\pi} \sin(\sqrt{\lambda - 1}\pi) = 0$$

Nontrivial solutions if $\sin(\sqrt{\lambda - 1}\pi) = 0$

In other words, when $\sqrt{\lambda - 1}$ is an integer

10.2.18

$$f(x) = \sin(4x) + 3\sin(6x) - \sin(10x)$$

$$c_4 = 1, c_6 = 3, c_10 = -1$$

$$u(x,t) = e^{-3\cdot 4^2 t}\sin(4x) + 3e^{-3\cdot 6^2 t}\sin(6x) - e^{-3\cdot 10^2 t}\sin 10x$$

10.2.20

$$f(x) = 0$$

$$g(x) = -2\sin(3x) + 9\sin(7x) - \sin(10x)$$

$$a_n = 0$$

$$3\cdot 3b_3 = -2, b_3 = \frac{-2}{9}$$

$$7\cdot 3b_7 = 9, \frac{9}{21}$$

$$u(x,t) = \frac{-2}{9}\sin(3\cdot 3t)\sin(3x) + \frac{9}{21}\sin(7\cdot 3t)\sin(7x) - \frac{1}{10}\sin(10\cdot 3t)\sin(10x)$$

10.3.2

$$\sin^2(-x) = \sin(-x)\sin(-x) = (-\sin(x))(-\sin(x)) = \sin^2(x)$$

It is even

10.3.6

$$x^{\frac{1}{5}} cos(x^2) = f(x)$$

$$f(-x) = (-x)^{\frac{1}{5}} cos(x^2)$$

It is neither

10.3.10

$$f(x) = \|x\|, -\pi < x < \pi$$

It is an even function. We know that even functions have Fourier series consisting of only consine functions, and the constant function $1 = \cos(0\pi x)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \|x\|\, dx$$

$$= \left[\frac{2}{\pi}\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\frac{\pi^2}{2} = \pi$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \|x\| \cos(nx)\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\cos(nx)\, dx$$

Using integration by parts, we obtain

$$\left[\frac{2}{\pi}\frac{\cos(nx) + nx\sin(nx)}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}(\cos(n\pi) - 1)$$

$$= 0 \text{ when n is even }, = \frac{-4}{\pi n^2} \text{ when n is odd}$$

$$f(x) = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{-4}{\pi(2k-1)^2}\cos(2k-1)x$$