Homework Set 6

4.3: 2, 4, 8, 30, 44, 50, 64, 66. **Chapter four:** 2, 6, 20, 41, 50, 66. **5.1**: 6, 10, 12, 14.

4.3.2 If we write the matrices with respect to the standard basis for $\mathbb{R}^{2\times 2}$, the question becomes: Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 6 \\ 8 \end{bmatrix}$ linearly independent? The vectors are linearly independent if and only if the kernel of $A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 6 \\ 1 & 4 & 7 & 8 \end{bmatrix}$ is $\{\vec{0}\}$. Since $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, A has nullity greater than zero and the matrices are not linearly independent. In fact, we can see that $\begin{bmatrix} 1 \\ -4 \\ 1 \\ 1 \end{bmatrix}$ is in the kernel, so $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - 4 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is an explicit demonstration of the linear dependence of the matrices.

4.3.4 With respect to the standard basis $\{1, t, t^2\}$ of P_2, the coordinates of $f(t), tf(t)$ and $g(t)$ are $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2k \\ 2+k \\ 1 \end{bmatrix}$ respectively. Thus they are a basis if the matrix $A = \begin{bmatrix} 1 & 0 & 2k \\ 1 & 1 & 2+k \\ 0 & 1 & 1 \end{bmatrix}$ is an isomorphism. $\text{Ref}(A) = \begin{bmatrix} 1 & 0 & 2k \\ 0 & 1 & 2-k \\ 0 & 0 & k-1 \end{bmatrix}$, so we will have a basis as long as k is not equal to 1.

4.3.8 $$\left[T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]_{\mathfrak{U}} = \left[\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\right]_{\mathfrak{U}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$
$$\left[T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right]_{\mathfrak{U}} = \left[\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right]_{\mathfrak{U}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\left[T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]_{\mathfrak{U}} = \left[\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}\right]_{\mathfrak{U}} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}.$$

So the matrix of T is $A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$. This is not an isomorphism. By inspection, a basis for the image is $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$, which corresponds to the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. After row-reducing A, we can see that $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for the kernel, which corresponds to the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. So the rank of T is 1.

4.3.30 $[T(1)]_{\mathfrak{U}} = \left[\frac{1-1}{h}\right]_{\mathfrak{U}} = [0]_{\mathfrak{U}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$[T(t)]_{\mathfrak{U}} = \left[\frac{t+h-t}{h}\right]_{\mathfrak{U}} = [1]_{\mathfrak{U}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$\left[T\left(t^2\right)\right]_{\mathfrak{U}} = \left[\frac{(t+h)^2-t^2}{h}\right]_{\mathfrak{U}} = [2t+h]_{\mathfrak{U}} = \begin{bmatrix} h \\ 2 \\ 0 \end{bmatrix}$.

So the matrix of T is $A = \begin{bmatrix} 0 & 1 & h \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. This is not an isomorphism. By inspection, a basis for the image is $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$, which corresponds to the polynomials $f(t) = 1, f(t) = t$. After row-reducing A, we can see that $\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$, is a basis for the kernel, which corresponds to f(t)=1. So the rank of T is 2. Geometrically, $T(f(t))$ is the function that takes values of t as input and returns the slope of the secant line between $f(t+h)$ and $f(t)$.

4.3.40 (a) The change of basis matrix is $S = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}$

(b) The matrices we "found" in Exercises 14 and 13 were $B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$. We compute $SB = \begin{bmatrix} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ and $AS = \begin{bmatrix} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$, so $AS = SB$.

4.3.50 $T(\cos(t)) = -\cos(t) - a\sin(t) + b\cos(t) = (b-1)\cos(t) - a\sin(t)$. $T(\sin(t)) = -\sin(t) + a\cos(t) + b\sin(t) = a\cos(t) + (b-1)\sin(t)$. So the matrix of this transformation with respect to the given basis is $\begin{bmatrix} b-1 & a \\ -a & b-1 \end{bmatrix}$. This is an isomorphism as long as $(b-1)^2 + a^2 \neq 0 \Leftrightarrow b \neq 1$ or $a \neq 0$.

4.3.64 a. $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = I_2$.
$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = P$.
$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = P^2 = \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix}$. So the matrix for this transformation is $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 8 \\ 1 & 3 & 9 \end{bmatrix}$.

b. $\text{Rref}(A) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$. So a basis for the image is given by the first and second columns of A, which correspond to I and P. A basis for the kernel is $\begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$ which corresponds to $\begin{bmatrix} 3 & -4 \\ 0 & 1 \end{bmatrix}$.

4.3.66 a. $T(x_1^2) = 2x_1x_2$. $T(x_1x_2) = x_2^2 - x_1^2$. $T(x_2^2) = -2x_1x_2$. So the matrix for the transformation is $A = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}$.

b. $\text{Rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So the first and second columns of A form a basis for the image. This basis corresponds to x_1^2, x_1x_2. A basis for the kernel is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ which corresponds to $x_1^2 + x_2^2$.

4.2 True, by the definition of kernel.

4.6 True, since the domain and codomain have the same finite dimension.

4.20 True. Any two vector spaces of the same dimension are isomorphic, and both of these spaces have dimension 10.

4.41 True. Consider the equation $a_1f_1 + \cdots + a_nf_n = 0$. Then since only f_n has a non-zero coefficient of x^n, $a_n = 0$. Then only f_{n-1} has a non-zero coefficient of x^{n-1}, so $a_{n-1} = 0$. Continuing in this fashion, we'll find that all of the a_i are zero, so the f_i are linearly independent.

4.50 True. An example is $\left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \middle| a, b \text{ in } \mathbb{R} \right\}$

4.66 False. If we call our basis $\mathfrak{B}, T(f) = 5f$ implies that $\begin{bmatrix} 3 & 5 \\ 0 & 4 \end{bmatrix} [f]_{\mathfrak{B}} = 5[f]_{\mathfrak{B}} \Rightarrow \left(\begin{bmatrix} 3 & 5 \\ 0 & 4 \end{bmatrix} - 5I_2 \right) [f]_{\mathfrak{B}} = \vec{0}$. But $\begin{bmatrix} 3 & 5 \\ 0 & 4 \end{bmatrix} - 5I_2 = \begin{bmatrix} -2 & 5 \\ 0 & -1 \end{bmatrix}$ is invertible, so f must be zero.

5.1.6 $\dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \dfrac{2 - 3 + 8 - 10}{\sqrt{1+1+4+4}\sqrt{4+9+16+25}} = \dfrac{-3}{\sqrt{10}\sqrt{54}} = \dfrac{-3}{6\sqrt{3}\sqrt{5}} = \dfrac{-\sqrt{3}}{6\sqrt{5}}$. So the angle between the vectors is $\cos^{-1}\left(\dfrac{-\sqrt{3}}{6\sqrt{5}}\right)$.

5.1.10 We wish to find those values of k such that $\vec{u} \cdot \vec{v} = 0$. This occurs when $2 + 3k + 4 = 0 \Rightarrow 3k = -6 \Rightarrow k = -2$.

5.1.12 $\|\vec{v}+\vec{w}\|^2 = (\vec{v}+\vec{w}) \cdot (\vec{v}+\vec{w}) = \vec{v} \cdot \vec{v} + 2\vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w} = \|\vec{v}\|^2 + 2\vec{v} \cdot \vec{w} + \|\vec{w}\|^2 \leq \|\vec{v}\|^2 + 2\|\vec{v}\|\|\vec{w}\| + \|\vec{w}\|^2$ (by Cauchy-Schwarz) $= (\|\vec{v}\| + \|\vec{w}\|)^2$. Taking square roots of both sides gives $\|\vec{v} + \vec{v}\| \leq \|\vec{v}\| + \|\vec{w}\|$.

5.1.14 Leonardo was wrong. Since the system is at rest, the horizontal components of the tensions must cancel. This means $\|\vec{F_1}\| \sin(\beta) = \|\vec{F_2}\| \sin(\alpha) \Rightarrow \frac{\|\vec{F_1}\|}{\|\vec{F_2}\|} = \frac{\sin(\alpha)}{\sin(\beta)}$. We can also see that $\tan(\beta) = \frac{\overline{EB}}{\overline{ED}}$ and $\tan(\alpha) = \frac{\overline{EA}}{\overline{ED}}$, so $\frac{\overline{EA}}{\overline{EB}} = \frac{\tan(\alpha)}{\tan(\beta)}$. Since $\frac{\sin(\alpha)}{\sin(\beta)}$ is not usually equal to $\frac{\tan(\alpha)}{\tan(\beta)}$, Leonardo's formula is not correct. (It is only correct when $\alpha = \beta$ if we restrict to $(0, \pi/2)$).