Homework Set 12

4.4: 2, 4, 6, 10, 16, 20, 28, 30, 36. **4.5**: 2, 6, 10, 14, 20, 24, 34, 40, 42.

4.4.2 The method can be applied. The right hand side of the equation is of the form $t^m e^{\alpha t} \cos \beta t$ with $m = 3, \alpha = 0, \beta = 4$.

4.4.4 The method can be applied since the right hand side can be written $e^{-4x} \sin x$.

4.4.6 The method can be applied since the right hand side can be written $4x$ (since $\sin^2 x + \cos^2 x = 1$).

4.4.10 We guess $y_p = A$. Then $y_p'' + 2y_p' - y_p = -A$. So we need $-A = 10$. Thus $y_p = -10$ is a particular solution.

4.4.16 The auxiliary equation is $r^2 - 1 = 0$ which has roots ± 1. Since the left hand side of the equation is $te^{\alpha t} \sin \beta t$, with $\alpha = 0$ and $\beta = 1$, we see $\alpha \pm i\beta = \pm i$ is not a root of the auxiliary equation. So by the method of undetermined coefficients, we guess $y_p = (A_1 t + A_0) \sin t + (B_1 t + B_0) \cos t$. Then $y_p'' - y_p = (-2A_1 t + (-2A_0 - 2B_1)) \sin t + (-2B_1 t + (-2B_0 + 2A_1)) \cos t \Rightarrow A_1 = -1/2, B_1 = 0, A_0 = 0, B_0 = -1/2$. So a particular solution is $y_p(t) = -\frac{1}{2} t \sin t - \frac{1}{2} \cos t$.

4.4.20 The auxiliary equation is $r^2 + 4 = 0$ which has roots $\pm 2i$. The right hand side of the equation is of the form $Ct^m e^{\alpha t} \sin \beta t$, where $m = 1, \alpha = 0$ and $\beta = 2$. Since $\alpha \pm i\beta = \pm 2i$ is a root of the auxiliary equation, we guess $y_p = t(A_1 t + A_0) \sin 2t + t(B_1 t + B_0) \cos 2t$. After setting $y_p'' + 4y_p = 16 \sin 2t$, the resulting system of equations gives $A_1 = 0, A_0 = 1, B_1 = -2, B_0 = 0$. So $y_p(t) = t \sin 2t - 2t^2 \cos 2t$ is a particular solution to the differential equation.

4.4.28 The auxiliary equation is $r^2 + 3r - 7 = 0$, which doesn't have rational roots, so 1 is not a root. Thus the form of the particular solution is $(A_4 t^4 + A_3 t^3 + A_2 t^2 + A_1 t + A_0) e^t$.

4.4.30 The auxiliary equation is $r^2 + 2r + 1 = 0$ which has double root -1. Thus $\alpha + i\beta = 1 + i$ is not a root of the auxiliary equation, so the form of the particular solution is $y_p = Ae^t \sin t + Be^t \cos t$.

4.4.36 The auxiliary equation is $r^4 - 3r^2 - 8 = 0$. We can check that i is not a root of this equation since $i^4 - 3i^2 - 8 = 1 + 3 - 8 = -4$. So we guess a solution of the form $y_p = A \sin t + B \cos t$. We find that $y_p^{(4)} - y_p'' - 8y_p = -4A \sin t - 4B \sin t$. We need $A = -\frac{1}{4}, B = 0$. So $y_p(t) = -\frac{1}{4} \sin t$ is a particular solution.

4.5.2 1. $y_p = t/4 - 1/8 + (1/4) \sin 2t$

2. $y_p = t/2 - 1/4 - (3/4)\sin 2t$
3. $y_p = (11/4)t - 11/8 - 3\sin 2t$.

4.5.6 The general solution to the homogeneous problem is $y_c = C_1e^{-2x}+C_2e^{-3x}$, so the general solution to the nonhomogeneous equation is $y_c + y_p = C_1e^{-2x} + C_2e^{-3x} + e^x + x^2$.

4.5.10 It can be applied since the right hand side can be written $(e^t+t)^2 = e^{2t} + 2te^t + t^2$ which is a sum of functions that we can solve using undetermined coefficients.

4.5.14 It can be applied since $\cosh t = \frac{e^t+e^{-t}}{2} = \frac{e^t}{2} + \frac{e^{-t}}{2}$, which is a sum of functions that we can solve using undetermined coefficients.

4.5.20 The auxiliary equation is $r^2 + 4 = 0$, which has roots $\pm 2i$. So the general solution is $y_c = C_1 \sin 2t + C_2 \cos 2t$. Now we find a particular solutions by guessing $y_p = A\sin t + B\cos t$ (since i is not a root of the auxiliary equations). Solving for A and B, we find $A = 1/3, B = -1/3$, so the general solution is $y_c + y_p = C_1 \sin 2t + C_2 \cos 2t + \frac{1}{3}\sin t - \frac{1}{3}\cos t$.

4.5.24 The auxiliary polynomial is $r^2 = 0$ which has double root 0. So $y_c = C_1 + C_2t$ is the general solution to the homogeneous problem. We can see that $y_p = t^3$ is a particular solution. So the general solution is $y = y_c+y_p = t^3 + C_1 + C^2t$. Now, $y(0) = 3$ implies $C_1 = 3$ and $y'(0) = -1$ implies $C_1 = -1$, so the solution to the initial value problem is $y(t) = t^3 - t + 3$.

4.5.34 The auxiliary equation has roots $-2, -3$. Thus $i, 2i$ are not roots, so the form of a particular solution is $y_p = A\sin t + B\sin t + C\sin 2t + D\cos 2t$.

4.5.40 First we find a solution to $y^{(4)} - 3y''' + 3y'' - y' = 20$. By inspection we see that $y_1 = 20t$ will work. Next we find a solution to $y^{(4)} - 3y''' + 3y'' - y' = 6t$. It is clear we'll need a polynomial of degree 2, so we guess $y_2 = At^2+Bt+C$. We get that $y_2^{(4)} - 3y_2''' + 3y_2'' - y_2' = 6A - (2At+B)$, so we need $A = -3$ and $B = -18$. Then by superposition $y_1+y_2 = -3t^2+2t$ is a particular solution to the original equation.

4.5.42
1. The auxiliary equation is $mr^2+br+k = 0$ which has roots $\frac{-b\pm\sqrt{b^2-4mk}}{2m} = \frac{-b\pm i\sqrt{4mk-b^2}}{2m} = \frac{-b}{2m}\pm i\frac{\sqrt{4mk-b^2}}{2m}$ (since $4m-b^2 > 0$). So the general solution to the homogeneous equation is $y_c = C_1e^{-\frac{b}{2m}t}\sin\frac{\sqrt{4mk-b^2}}{2m}t + C_2e^{-\frac{b}{2m}t}\cos\frac{\sqrt{4mk-b^2}}{2m}t$. Now, when $g(t) = \sin t$, we use the method of undetermined coefficients to find a particular solution. No matter what β is, $i\beta$ is not a solution to the auxiliary equation since the roots of the auxiliary equation have non-zero real part (since $b, m > 0$). Thus the form of the particular solution is $y_p = A\sin\beta t + B\cos\beta t$. So the general solution is $y(t) = C_1e^{-\frac{b}{2m}t}\sin\frac{\sqrt{4mk-b^2}}{2m}t + C_2e^{-\frac{b}{2m}t}\cos\frac{\sqrt{4mk-b^2}}{2m}t + A\sin\beta t + B\cos\beta t$ (where A, B are fixed constants depending on m, b, k).

2. As $t \to +\infty$, $C_1 e^{-\frac{b}{2m}t} \sin \frac{\sqrt{4mk-b^2}}{2m} t + C_2 e^{-\frac{b}{2m}t} \cos \frac{\sqrt{4mk-b^2}}{2m} t$ will tend to zero, since the coefficient in the exponent (i.e. $-\frac{b}{2m}$) is negative. Thus eventually the solution will look like $A \sin \beta t + B \cos \beta t$. One interpretation of this is that the motion of the spring will eventually become in synch with the external force in the sense that it will have the same period.