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Math54 Sample Midterm I Selected Solutions, Fall 2007

This is a closed book, closed notes exam. You need to justify every one of your answers unless you are asked not to do so. Completely correct answers given without justification will receive little credit. Look over the whole exam to find problems that you can do quickly. You need not simplify your answers unless you are specifically asked to do so. Hand in this exam before you leave.

Problem	Maximum Score	Your Score
1	5	
2	19	
3	19	
4	19	
5	19	
6	19	
Total	100	

1. (5 Points)

Your Name: ______________________________

Your GSI: ______________________________

Your SID: ______________________________

2. (19 Points)

   (a) Solve linear system of equations $A\,x = b$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 2 & 1 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 6 \\ 7 \\ 4 \end{pmatrix}.$$

   (b) Consider linear system of equations $A\,x = b$, where

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & k^2 \\ 2 & 1 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 6 \\ 3k \\ 4 \end{pmatrix}.$$

For what values of k does the system have a unique solution? infinite number of solutions? no solution?

Solution: Perform `rref` on the augmented matrix to get

$$\begin{pmatrix} 1 & 2 & 4 & 6 \\ 0 & 1 & 7/3 & 11/3 \\ 0 & 0 & k^2-4 & 3(k-2) \end{pmatrix}.$$

This system has unique solution for $k \neq \pm 2$, infinite number of solutions for $k = 2$, and no solutions for $k = -2$.

3. (19 Points) Let $\mathcal{P}$ be the set of all functions of the form $c_0 + c_1 \sin(x)\cos(x) + c_2 \cos^2(x) + c_3 \sin^2(x)$, where the c's are arbitrary real constants. It is known that $\mathcal{P}$ is a linear space under the usual function addition and scalar multiplication. Find the dimension and a basis for $\mathcal{P}$.

   Solution: We observe that

   $$\sin(x)\cos(x) = 1/2\sin(2x), \quad \cos^2(x) = (1+\cos(2x))/2, \quad \sin^2(x) = (1-\cos(2x))/2.$$

   Hence

   $$c_0+c_1\sin(x)\cos(x)+c_2\cos^2(x)+c_3\sin^2(x) = (c_0+(c_2+c_3)/2)+(c_1/2)\sin(2x)+(c_2-c_3)/2\cos(2x),$$

   which is a linear combination of 1, $\sin(2x)$ and $\cos(2x)$, which are linearly independent. Hence the dimension is 3 with 1, $\sin(2x)$ and $\cos(2x)$ as a basis.

4. (19 Points) Let $u_1, \cdots, u_m$ be vectors in $\mathbf{span}\{v_1, \cdots, v_k\}$; and let $v_1, \cdots, v_k$ be vectors in $\mathbf{span}\{w_1, \cdots, w_n\}$. Show that $u_1, \cdots, u_m$ are vectors in $\mathbf{span}\{w_1, \cdots, w_n\}$.

   Solution: For each $1 \le j \le k$, since $v_j \in \mathbf{span}\{w_1, \cdots, w_n\}$, there exists a vector c_j so that $v_j = (w_1, \cdots, w_n)\, c_j$. Hence

$$(v_1, \cdots v_k) = (w_1, \cdots, w_n)\,(c_1, \cdots, c_k)\,.$$

   For each $1 \le i \le m$, since $u_i \in \mathbf{span}\{v_1, \cdots, v_k\}$, there exists a vector b_i so that $u_i = (v_1, \cdots, v_k)\, b_i$. Hence

$$u_i = (v_1, \cdots, v_k)\, b_i = (w_1, \cdots, w_n)\,(c_1, \cdots, c_k)\, b_i.$$

   Define $a_i = (c_1, \cdots, c_k)\, b_i$, then

$$u_i = (w_1, \cdots, w_n)\, a_i,$$

   which means u_i is a linear combination of $w_1, \cdots, w_n$, and hence a vector in $\mathbf{span}\{w_1, \cdots, w_n\}$.

5. (19 Points) If the image of an $n \times n$ matrix A is $\mathbf{R}^n$, show that A must be invertible.

   Proof: Since

   $$\mathbf{dim}(\mathbf{Ker}(A)) + \mathbf{dim}(\mathbf{Im}(A)) = n,$$

   and since $\mathbf{dim}(\mathbf{Im}(A)) = n$, it follows that $\mathbf{dim}(\mathbf{Ker}(A)) = 0$. Hence A must be invertible.

6. (19 Points) Find examples of $n \times n$ matrices A and B such that A, B are not invertible but $A + B$ is.

Solution: Let $A = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix}$ and B be the matrix that is zero everywhere except the $(1, 1)$ position, which is 1. Then both A and B are singular, but $A + B$ is the identity matrix, which is invertible.