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Math54 Sample Midterm II, Fall 2007

This is a closed everything exam, except a standard one-page cheat sheet (on one-side only). You need to justify every one of your answers. Completely correct answers given without justification will receive little credit. Problems are not necessarily ordered according to difficulties. You need not simplify your answers unless you are specifically asked to do so.

Problem	Maximum Score	Your Score
1	5	
2	19	
3	19	
4	19	
5	19	
6	19	
Total	100	

1. (5 Points) Write your personal information below.

   Your Name: ______________________________

   Your GSI: ______________________________

   Your SID: ______________________________

2. Consider the following map from $\mathbf{R}^2$ to $\mathbf{R}^2$:

$$\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y-1 \\ x-y+1 \end{pmatrix}.$$

Is $\mathbf{T}$ a linear transform? Explain.

Solution: No. For constant $c = 0$,

$$\mathbf{T}\left(c * \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \mathbf{T}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ +1 \end{pmatrix} \neq c * \mathbf{T}\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

3. Is the following identity true for all pairs of invertible matrices $A, B \in \mathcal{R}^{n\times n}$?

$$\left((A\,B)^{-1}\right)^T = \left(A^{-1}\right)^T \left(B^{-1}\right)^T.$$

Explain your answer.

Solution: Yes.

$$\left((A\,B)^{-1}\right)^T = \left(B^{-1}\,A^{-1}\right)^T = \left(A^{-1}\right)^T \left(B^{-1}\right)^T.$$

4. Give an explicit formula for the components of the vector

$$\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for any integer $k > 0$.

Solution: Eigenvalues are -5 and 5, with respective eigenvectors $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Let $S = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$. Then $S^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}/5$ and

$A^k = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} (-5)^k & \\ & 5^k \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}/5 = \begin{pmatrix} 5^{k-1} - 4*(-5)^{k-1} & 2*5^{k-1} + 2*(-5)^{k-1} \\ 2*5^{k-1} + 2*(-5)^{k-1} & 4*5^{k-1} - (-5)^{k-1} \end{pmatrix}$

5. Show that the following function

$$\mathbf{x} \cdot \mathbf{y} = 3x_1y_1 + 2x_2y_2$$

is an inner product on $\mathbf{R}^2$.

Proof:

- $$\mathbf{x} \cdot \mathbf{y} = 3x_1y_1 + 2x_2y_2 = \mathbf{y} \cdot \mathbf{x}.$$

- For any fixed $\mathbf{y}$, $\mathbf{x} \cdot \mathbf{y}$ is clearly a linear function in $\mathbf{x}$. Hence the conditions

  $$(\mathbf{f} + \mathbf{g}) \cdot \mathbf{y} = \mathbf{f} \cdot \mathbf{y} + \mathbf{g} \cdot \mathbf{y}$$

  and

  $$(\alpha * \mathbf{f}) \cdot \mathbf{y} = \alpha * (\mathbf{f} \cdot \mathbf{y})$$

  both hold for vectors $\mathbf{f}$ and $\mathbf{g}$ and scalar α.

- $$\mathbf{x} \cdot \mathbf{x} = 3x_1^2 + 2x_2^2 \geq 0$$

  for any $\mathbf{x}$. In addition, the right hand side is zero only when both x_1 and x_2 are zero, which means $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = 0$. Hence this is an inner product.

6. Let

$$V = \textbf{span}\left(\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}\right).$$

Find an orthonormal basis for the orthogonal complement of V.

Solution: The orthogonal complement of V is $\textbf{Ker}(V^T)$. Since

$$\textbf{rref}(V^T) = \begin{pmatrix}1 & 0 & 0 & 1\\0 & 1 & 1 & 0\end{pmatrix},$$

A basis for $\textbf{Ker}(V^T)$ is $\{\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\}$. These two basis vectors are mutually orthogonal.

To make the basis orthonormal, we normalize them to get $\{\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}/\sqrt{2}, \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}/\sqrt{2}\}$.