Solutions HW 7

5.1.16 Consider the vectors

$$u_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \quad u_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \quad u_3 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

in $\mathbb{R}^4$. Can you find a vector u_4 such that u_1, u_2, u_3, u_4 are orthonormal? If so, how many such vectors are there?

Note that u_1, u_2, u_3 are already orthonormal, so we just need to find u_4, which must satisfy $u_1 \cdot u_4 = u_2 \cdot u_4 = u_3 \cdot u_4 = 0$ and $u_4 \cdot u_4 = 1$. If $u_4 = (a, b, c, d)^T$, then the first three conditions give the linear equations

$$\begin{aligned} \tfrac{1}{2}a + \tfrac{1}{2}b + \tfrac{1}{2}c + \tfrac{1}{2}d = 0 \\ \tfrac{1}{2}a + \tfrac{1}{2}b - \tfrac{1}{2}c - \tfrac{1}{2}d = 0 \\ \tfrac{1}{2}a - \tfrac{1}{2}b + \tfrac{1}{2}c - \tfrac{1}{2}d = 0 \end{aligned}$$

Row reducing these equations leads to the system

$$\begin{aligned} a + b + c + d = 0 \\ b + d = 0 \\ c + d = 0 \end{aligned}$$

which has general solution $u_4 = (a, b, c, d)^T = (t, -t, -t, t)^T$. Any such vector has dot product zero with u_1, u_2, u_3. For an orthonormal basis, we want $u_4 \cdot u_4 = 1$, which gives $t = \pm\frac{1}{2}$, so there are 2 choices for u_4, namely

$$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \; or \; \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

5.1.20 (See the book for the full statement of the problem, which involves finding a least squares estimate for a line $y = mx$).

Let $\overline{x}$ be the vector of x coordinates and $\overline{y}$ be the vector of y coordinates. Geometrically, $\{m\overline{x})\}$ is a 1 dimension vector space. We which to minimize the distance from y into this space; this is accomplished when $m\overline{x}$ is the projection of y into this space. Note that $\frac{x}{||x||}$ is a unit vector in this space, so the corresponding projection should be $(y \cdot \frac{x}{||x||})\frac{x}{||x||} = \frac{y \cdot x}{||x||^2}x$. In particular, we should take $m = \frac{y \cdot x}{||x||^2} = \frac{4182.9}{198.53^2} = 0.106\ldots$.

$r = \frac{m||x||}{||y||}$ from our formula. (Note that $||y||/||x||$ is the expected slope if all of the points did lie on a line, so r can be thought of as a ratio of two slopes.) If you draw the relevant line in the picture, most points will be near the line (even if none are on the line).

5.1.22 Consider a basis $v_1, v_2, \dots, v_m$ of a subspace V of $\mathbb{R}^n$. Show that a vector x in $\mathbb{R}^n$ is orthogonal to v if and only if it is orthogonal to all the vectors $v_1, \dots, v_n$.

If x is orthogonal to all the vectors $v_1, \dots, v_n$, then for any $v \in V$, we can write $v = c_1 v_1 + \dots c_m v_m$. Then $x \cdot v = x \cdot (c_1 v_1 + \dots c_m v_m) = c_1(x \cdot v_1) + \dots + c_m(x \cdot v_m) = 0$ (since each $x \cdot v_i = 0$). So x is orthogonal to every vector in V.

If $x \cdot v = 0$ for all $v \in V$, then this applies in particular to the vectors $v_1, \dots v_m \in V$.

5.1.24 Complete the proof of Fact 5.1.4: Orthogonal Projection are linear transformations.

We must show that $proj_V(x + y) = proj_V(x) + proj_V(y)$ and $proj_V(cx) = cproj_V(x)$.

From page 189, if $u_1, \dots u_m$ are an orthonormal basis for V, then $proj_V(x + y) = (u_1 \cdot (x+y))u_1 + \dots + (u_m \cdot (x+y))u_m = ((u_1 \cdot x) + (u_1 \cdot y))u_1 + \dots + ((u_m \cdot x) + (u_m \cdot y))u_m = [(u_1 \cdot x)u_1 + \dots + (u_m \cdot x)u_m] + [(u_1 \cdot y)u_1 + \dots + (u_m \cdot y)u_m] = proj_V(x) + proj_V(y)$.

Similarly $proj_V(cx) = (u_1 \cdot cx)u_1 + \dots + (u_m \cdot cx)u_m = c[(u_1 \cdot x)u_1 + \dots + (u_m \cdot x)u_m] = cproj_V(x)$.

5.1.30 Consider a subspace V or $\mathbb{R}^n$ and a vector x in $\mathbb{R}^n$. Let $y = proj_V(x)$. What is the relation ship between $||y||^2$ and $y \cdot x$?

The two quantities are equal. One can see this by writing $x = y + w$; then y is the parallel component to V and w is orthogonal to V, and in particular $y \cdot w = 0$. Then $x \cdot y = (y + w) \cdot y = y \cdot y = ||y||^2$ as desired.

5.2.2 Perform Gram-Schmidt on the vectors

$$\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix}$$

We have

$$u_1 = \begin{pmatrix} 6/7 \\ 3/7 \\ 2/7 \end{pmatrix}$$

$$v_2^{\perp} = \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix} -0 \begin{pmatrix} 6/7 \\ 3/7 \\ 2/7 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 2/7 \\ -6/7 \\ 3/7 \end{pmatrix}$$

5.2.14 Perform Gram-Schmidt on the vectors

$$\begin{pmatrix} 1 \\ 7 \\ 1 \\ 7 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 7 \\ 2 \\ 7 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 8 \\ 1 \\ 6 \end{pmatrix}$$

We have

$$u_1 = \begin{pmatrix} 1/10 \\ 7/10 \\ 1/10 \\ 7/10 \end{pmatrix}$$

$$v_2^{\perp} = \begin{pmatrix} 0 \\ 7 \\ 2 \\ 7 \end{pmatrix} -10 \begin{pmatrix} 1/10 \\ 7/10 \\ 1/10 \\ 7/10 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$v_3^{\perp} = \begin{pmatrix} 1 \\ 8 \\ 1 \\ 6 \end{pmatrix} -10 \begin{pmatrix} 1/10 \\ 7/10 \\ 1/10 \\ 7/10 \end{pmatrix} -0 \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$u_3 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ -1\sqrt{2} \end{pmatrix}$$

5.2.16 Find the QR decomposition of the matrix

$$\begin{pmatrix} 6 & 2 \\ 3 & -6 \\ 2 & 3 \end{pmatrix}$$

From the work in problem 2, we can take Q to be

$$\begin{pmatrix} 6/7 & 2/7 \\ 3/7 & -6/7 \\ 2/7 & 3/7 \end{pmatrix}$$

R is calculated by computing various dot products of the column vectors of Q with the column vectors of the original matrix, so we get R to be

$$\begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$$

5.2.28 Find the QR decomposition of the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 7 & 7 & 8 \\ 1 & 2 & 1 \\ 7 & 7 & 6 \end{pmatrix}$$

From the work in problem 2, we can take Q to be

$$\begin{pmatrix} 1/10 & -1/\sqrt{2} & 0 \\ 7/10 & 0 & 1/\sqrt{2} \\ 1/10 & 1/\sqrt{2} & 0 \\ 7/10 & 0 & -1/\sqrt{2} \end{pmatrix}$$

R is calculated by computing various dot products of the column vectors of Q with the column vectors of the original matrix, so we get R to be

$$\begin{pmatrix} 10 & 10 & 10 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

5.2.32 Find an orthonormal basis of the plane $x_1 + x_2 + x_3 = 0$.

First we find a basis for the plane by backsolving the equation. For example, one such basis is

$$v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Next we apply Gram-Schmidt to this basis to make it orthonormal.

$$u_1 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

$$v_2^\perp = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad -1/\sqrt{2} \quad \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1 \\ -1/2 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} -1/\sqrt{6} \\ \sqrt{2}/\sqrt{3} \\ -1/\sqrt{6} \end{pmatrix}$$

u_1, u_2 form an orthonormal basis for the plane $x_1 + x_2 + x_3 = 0$.

5.3.2 Is

$$\begin{pmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{pmatrix}$$

orthogonal?

Yes. Either one can note that the columns are orthogonal vectors, or one can compute $A^T A$ and see that you get the identity matrix.

5.3.10 If A and B are orthogonal matrices, is $B^{-1}AB$ orthogonal also?

Yes. By fact 5.3.4 a, B^{-1} is also orthogonal, and then applying Fact 5.3.4 b several times shows that the product $B^{-1}AB$ of three orthogonal matrices is an orthogonal matrix. (Alternatively, one can show $(B^{-1}AB)^T = (B^{-1}AB)^{-1}$ directly.)

5.3.20 If A and B are symmetric, is AB^2A symmetric?

Yes. $(AB^2A)^T = A^T(B^2)^T A^T = AB^2A$ as required.

5.3.22 If B is an $n \times n$ matrix, is BB^T symmetric?

Yes. $(BB^T)^T = (B^T)^T B^T = BB^T$ as desired.

5.3.40 Consider the subspace W of $\mathbb{R}^4$ spanned by the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 9 \\ -5 \\ 3 \end{pmatrix}$$

Find the matrix of orthogonal projection onto W.

We start by finding an orthonormal basis of W using Gram-Schmidt. We get

$$u_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$v_2^{\perp} = \begin{pmatrix} 1 \\ 9 \\ -5 \\ 3 \end{pmatrix} - 4 \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ -7 \\ 1 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} -1/10 \\ 7/10 \\ -7/10 \\ 1/10 \end{pmatrix}$$

So setting

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{-1}{10} \\ \frac{1}{2} & \frac{7}{10} \\ \frac{1}{2} & \frac{-7}{10} \\ \frac{1}{2} & \frac{1}{10} \end{pmatrix}$$

We have that the matrix of orthogonal projection onto W is QQ^T or

$$\begin{pmatrix} \frac{26}{100} & \frac{18}{100} & \frac{32}{100} & \frac{24}{100} \\ \frac{18}{100} & \frac{74}{100} & \frac{-24}{100} & \frac{32}{100} \\ \frac{32}{100} & \frac{-24}{100} & \frac{74}{100} & \frac{18}{100} \\ \frac{24}{100} & \frac{32}{100} & \frac{18}{100} & \frac{26}{100} \end{pmatrix}$$

5.3.54 Find the dimension of the space of all skew symmetric matrices.

The dimension is $\frac{n^2-n}{2}$, since such a matrix must have all 0's on the diagonal, and then is determined by the values in the strictly upper triangular part of the matrix.