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Math54 Sample Final Exam Solutions

This is a closed everything exam, except a standard one-page cheat sheet (on one-side only). You need to justify every one of your answers. Completely correct answers given without justification will receive little credit. Problems are not necessarily ordered according to difficulties. You need not simplify your answers unless you are specifically asked to do so.

Problem	Maximum Score	Your Score
1	4	
2	16	
3	16	
4	16	
5	16	
6	16	
7	16	
Total	100	

1. (5 Points) Write your personal information below.

Your Name: ______________________________

Your GSI: ______________________________

Your SID: ______________________________

2. Find bases for the image and kernel for

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

3. Let

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

   (a) Find the eigenvalues and corresponding eigenvectors of A.

   (b) Diagonalize A.

4. Consider the family of boundary value problems

$$\begin{aligned} x'' + k^2 x &= 0, \\ x(0) &= 0, \\ x(1) &= 1. \end{aligned}$$

(a) Solve the boundary value problem for $k = 2$.

Solution: General solution to the DE is $x(t) = A\cos 2t + B\sin 2t$, where A and B are constants. The boundary condition $x(0) = 0$ implies $A = 0$, and the boundary condition $x(1) = 1$ imples $B = 1/\sin 2$. Hence the solution is $x(t) = \sin 2t/\sin 2$.

(b) For which values of k does the boundary value problem have **no** solutions?

Solution: As before, the general solution to the DE is $x(t) = A\cos 2t + B\sin 2t$, where A and B are constants. The boundary condition $x(0) = 0$ implies $A = 0$. However, the boundary condition $x(1) = 1$ now imples $B\sin k = 1$, which has a solution if and only if $\sin k \neq 0$. Hence the boundary value problem has no solutions if and only if $\sin k = 0$, or that k is an integral multiple of π.

5. Let u and v be vectors in $\mathcal{R}^n$. Form the matrix

$$G = \begin{pmatrix} \left(u^T u\right) & \left(u^T v\right) \\ \left(v^T u\right) & \left(v^T v\right) \end{pmatrix}.$$

Show that u and v are linearly independent if and only if G is invertible.

Solution: We use the Cauchy-Schwartz inequality. We assume that u and v are non-singular (otherwise G is singular and u and v are linearly dependent.)

If u and v are linearly dependent, then there exists a constant k such that $u = kv$. Replacing u by kv in the matrix G, it is easy to see that the determinant of G is zero, hence G is not invertible. This shows that if u and v are linearly dependent then G is not invertible.

On the other hand, assume that G is not invertible, so that the determinant of G is zero. This implies

$$\left(u^T u\right)\left(v^T v\right) - \left(u^T v\right)^2 = 0,$$

or that

$$u^T u - \frac{\left(u^T v\right)^2}{v^T v} = 0.$$

Some simple algebra now shows that

$$\left(u - \frac{u^T v}{v^T v} v\right)^T \left(u - \frac{u^T v}{v^T v} v\right) = u^T u - \frac{\left(u^T v\right)^2}{v^T v} = 0.$$

Hence $u - \frac{u^T v}{v^T v} v = 0$, which shows that u and v must be linearly dependent.

To summarize, we have shown that u and v are linearly dependent if and only if G is not invertible. In other words, u and v are linearly independent if and only if G is invertible.

6. Let q_1, $q_2, ..., q_k$ be a set of orthonormal vectors in $\mathcal{R}^n$. Show that there exists vectors q_{k+1}, $q_{k+2}, ..., q_n$ so that
$$Q = (q_1 \quad q_2 \cdots \quad q_n)$$
is an orthogonal matrix.

Solution: Since q_1, $q_2, ..., q_k$ are a set of orthonormal vectors, they must be linearly independent. Thus, they can be extended into a basis for $\mathcal{R}^n$. Let the extended basis be q_1, $q_2, ..., q_k$, $w_{k+1}, ..., w_n$. Now perform Gram-Schmidt process on these vectors to get $QR = (q_1, q_2, ..., q_k, w_{k+1}, ..., w_n)$. Q is now an orthogonal matrix.

However, since $q_1, q_2, ..., q_k$ are orthonormal, the Gram-Schmidt process will keep them unchanged. In other words, the first k columns of Q are precisely $q_1, q_2, ..., q_k$.

7. Compute the Fourier cosine series of the function $f(x) = 1 - x$ on the interval $(0, \pi)$.