Math221 Midterm Solutions, 2007

1. On page 5 in the text, backward stability is defined as follows:

   > If $\mathbf{alg}(x)$ is our algorithm for $f(x)$, including the effects of roundoff, we call $\mathbf{alg}(x)$ a backward stable algorithm for $f(x)$ if for all x there is a "small" δx such that $\mathbf{alg}(x) = f(x + \delta x)$. δx is called the backward error. Informally, we say that we get the exact answer $(f(x + \delta x))$ for a slightly wrong problem $(x + \delta x)$.

   Identify the x and $f(x)$ for

   - systems of linear equations.
     Solution: $Ax = b$, where A and b are the input, the x in the definition, and $x = A^{-1}b$ is the $f(x)$, the output.
   - Least Squares problems.
     Solution: This problem is
     $$\min_x \|Ax - b\|_2,$$
     where A and b are the input, the x in the definition, and the solution to the normal equation
     $$A^T Ax = A^T b,$$
     is the $f(x)$, the output.
   - QR factorization.
     Solution: The QR factorization is
     $$A = QR,$$
     where Q is orthogonal and R is upper triangular. Here A is the input, the x in the definition, and Q and R are the output $f(x)$.

   In the first two cases, also identify a backward stable algorithm for computing $f(x)$. In the third case, explain why a backward stable algorithm might NOT exist. We exclude overflow/underflow considerations.

   Solution: GEPP is backward stable for solving linear systems of equations as long as the element growth factor is controled. QR factorization is a good method for solving both linear equations and least squares problems.

   Computing a backward stable QR factorization would be very tricky. This is because Q is an orthogonal matrix. By definition, even the Q factor in the perturbed output $f(x + \delta x)$ must still be exactly orthogonal. It is unlikely any algorithm will be able to do so in general.

2. Let
$$A = \begin{pmatrix} I & Z \\ 0 & I \end{pmatrix},$$
where $Z \in \mathbf{R}^{n\times n}$ and I is the identity matrix. Express the 2-norm condition number of A using the singular values of Z.

Solution: Let $Z = U\Sigma V^T$ be the SVD of Z, then
$$A = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} I & \Sigma \\ 0 & I \end{pmatrix} \begin{pmatrix} U^T & 0 \\ 0 & V^T \end{pmatrix}.$$
Of the three matrices on the right hand side, only the middle matrix is not orthogonal, and its singular values are the singular values of A. Each diagonal entry σ of Σ induces a pair of singular values in the 2×2 matrix
$$\begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix}.$$
These singular values are
$$\sqrt{\frac{2 + \sigma^2 \pm \sigma\sqrt{\sigma^2 + 4}}{2}}.$$
The singular values of A are all these pairs of singular values. Hence the 2-norm condition number of A is
$$\sqrt{\frac{2 + \sigma_{\max}^2 + \sigma_{\max}\sqrt{\sigma_{\max}^2 + 4}}{2 + \sigma_{\min}^2 + \sigma_{\min}\sqrt{\sigma_{\min}^2 + 4}}}.$$

3. Let T be an $m \times m$ lower triangular matrix and b be an m-vector. Show how to solve the $m \times m$ linear system of equations $Tx = b$ in about m^2 flops.

   Solution: Forward substitution.

4. Let $H \in \mathbf{R}^{5\times 5}$ be a matrix of the form

$$H = \begin{pmatrix} x & x & x & x & x \\ x & x & x & x & x \\ & x & x & x & x \\ & & x & x & x \\ & & & x & x \end{pmatrix},$$

where all the x's denote non-zero entries. Matrices with the non-zero pattern of H are called Hessenberg matrices. Show how to use 4 Givens rotations to QR factorize H.

Solution: A Givens rotation is of the form

$$G = \begin{pmatrix} c & s \\ -s & c \end{pmatrix},$$

where $c^2 + s^2 = 1$. It is used to zero out a component in a 2 dimensional vector. We apply a Givens rotation to zero out the $(2, 1)$ entry:

$$G_1 H = \begin{pmatrix} \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ & \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ & x & x & x & x \\ & & x & x & x \\ & & & x & x \end{pmatrix},$$

where the $\mathbf{x}$ entries represent entries that have been changed.

There are 4 subdiagonal entries in H. We use 4 Givens rotations, executed one by one, to zero out all of them.

5. Consider the following variant of the least squares problem:

$$\min_x \left(\|Ax - b\|_2^2 + \rho^2 \|x\|_2^2 \right),$$

where the matrix A, the vector b and scalar $\rho > 0$ are given. Find the normal equation for this problem and develop a QR-type algorithm for solving the normal equation.

Solution: We let

$$\widehat{A} = \begin{pmatrix} A \\ \rho I \end{pmatrix} \quad \text{and} \quad \widehat{b} = \begin{pmatrix} b \\ 0 \end{pmatrix}.$$

Then the given problem becomes

$$\min_x \|\widehat{A}x - \widehat{b}\|_2^2,$$

to which the standard QR factorization method can be applied.