Math221 Midterm, Oct. 18, 2007

1. On page 5 in the text, backward stability is defined as follows:

   > IF $\mathbf{alg}(x)$ IS OUR ALGORITHM FOR $f(x)$, INCLUDING THE EFFECTS OF ROUNDOFF, WE CALL $\mathbf{alg}(x)$ A backward stable algorithm FOR $f(x)$ IF FOR ALL x THERE IS A "SMALL" δx SUCH THAT $\mathbf{alg}(x) = f(x + \delta x)$. δx IS CALLED THE backward error. INFORMALLY, WE SAY THAT WE GET THE EXACT ANSWER $(f(x + \delta x))$ FOR A SLIGHTLY WRONG PROBLEM $(x + \delta x)$.

   Identify the x and $f(x)$ for

   - systems of linear equations.
   - Least Squares problems.
   - QR factorization.

   In the first two cases, also identify a backward stable algorithm for computing $f(x)$. In the third case, explain why a backward stable algorithm might NOT exist. We exclude overflow/underflow considerations.

2. Let
$$A = \begin{pmatrix} I & Z \\ 0 & I \end{pmatrix},$$
where $Z \in \mathbf{R}^{n \times n}$ and I is the identity matrix. Express the 2-norm condition number of A using the singular values of Z.

3. Let T be an $m \times m$ lower triangular matrix and b be an m-vector. Show how to solve the $m \times m$ linear system of equations $Tx = b$ in about m^2 flops.

4. Let $H \in \mathbf{R}^{5\times 5}$ be a matrix of the form

$$H = \begin{pmatrix} x & x & x & x & x \\ x & x & x & x & x \\ & x & x & x & x \\ & & x & x & x \\ & & & x & x \end{pmatrix},$$

where all the x's denote non-zero entries. Matrices with the non-zero pattern of H are called Hessenberg matrices. Show how to use 4 Givens rotations to QR factorize H.

5. Consider the following variant of the least squares problem:

$$\min_{x} \left(\|Ax - b\|_2^2 + \rho^2 \|x\|_2^2 \right),$$

where the matrix A, the vector b and scalar $\rho > 0$ are given. Find the normal equation for this problem and develop a QR-type algorithm for solving the normal equation.