Math 221 Final Exam, Dec. 12, 2007

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1. (20 Points) Let $A \in \mathbf{R}^{n\times n}$. Show that $\|A^k\| \leq \|A\|^k$ for all positive integers k and for any induced norm $\|\cdot\|$.

2. (20 Points) Assume that the matrix $A \in \mathbf{R}^{n\times n}$ is diagonalizable with real eigenvalues. In other words, there exists a diagonal matrix D and a non-singular matrix T, with $D, T \in \mathbf{R}^{n\times n}$, such that $A = TDT^{-1}$. Show that

   (a) The matrix TDT^T is symmetric.

   (b) There exist symmetric matrices $W, V \in \mathbf{R}^{n\times n}$, with W non-singular, so that $A = VW^{-1}$.

3. (20 Points) Let $W \in \mathbf{R}^{m \times m}$ be a non-singular matrix and let $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$, with $n \leq m$. Consider the weighted least squares problem $\min_x \|W(Ax - b)\|_2$.

   (a) Write the normal equation for this problem.

   (b) Sketch a backward stable algorithm for solving the problem. Count the number of flops up to the leading terms.

4. (20 Points) Let $A, Q \in \mathbf{R}^{n\times n}$ with $Q = \begin{pmatrix} q_1 & \cdots & q_n \end{pmatrix}$ orthogonal. Suppose that

$$Q^T A Q = H \tag{1}$$

is upper Hessenberg such that all subdiagonal entries of H are positive. The Implicit Q Theorem states that columns q_2 through q_n are uniquely determined by q_1. Derive the Arnoldi algorithm from (1).

5. (20 Points) Use the SVD to show that if $A \in \mathbf{R}^{m \times n}$ with $m \geq n$, then there exists $Q \in \mathbf{R}^{m \times n}$ with orthornormal columns ($Q^T Q = I$) and a positive semidefinite matrix $P \in \mathbf{R}^{n \times n}$ such that $A = QP$.