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Math221: Matrix Computations
Homework #11 Solutions

- **Solution to Problem 6.5:** We can write

$$A = X \Lambda_A X^{-1}, \quad B = Y \Lambda_B Y^{-1}.$$

By Lemma 6.3,

$$\begin{aligned} I \otimes A + B \otimes I &= I \otimes \left(X \Lambda_A X^{-1}\right) + \left(Y \Lambda_B Y^{-1}\right) \otimes I \\ &= \left(Y\, Y^{-1}\right) \otimes \left(X \Lambda_A X^{-1}\right) + \left(Y \Lambda_B Y^{-1}\right) \otimes \left(X\, X^{-1}\right) \\ &= \left(Y \otimes X\right) \left(I \otimes \Lambda_A + \Lambda_B \otimes I\right) \left(Y^{-1} \otimes X^{-1}\right) \\ &= \left(Y \otimes X\right) \left(I \otimes \Lambda_A + \Lambda_B \otimes I\right) \left(Y \otimes X\right) \left(Y \otimes X\right)^{-1}. \end{aligned}$$

The matrix $I \otimes \Lambda_A + \Lambda_B \otimes I$ is a diagonal matrix, with its diagonal entries being $\alpha_i + \beta_j$ for $1 \leq i, j \leq n$.

The Sylvester equation $AX + XB^T = C$ can be equivalently written as

$$\left(I \otimes A + B \otimes I\right) \mathbf{vec}(X) = \mathbf{vec}(C).$$

This equation is solvable for any given C if and only if $I \otimes A + B \otimes I$ is non-singular. In other words, if and only if $\alpha_i + \beta_j \neq 0$ for all $1 \leq i, j \leq n$. Since B and B^T have the same set of eigenvalues, this is the same set of conditions for the non-singularity of the alternative Sylvester equation $AX + XB = C$.