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Math221: Matrix Computations
Homework #10 Solutions

- **Problem 5.13 Solution:** The QR iteration takes the following form:

  Let $A_0 = A$, $i = 0$;
  repeat

  $A_i - \sigma_i I = Q_i R_i$;
  $A_{i+1} = R_i Q_i + \sigma_i I$;
  $i = i + 1$.

  The shift σ_i in the QR iteration is chosen to be the (n, n) entries of A_i. In other words, $\sigma_i = e_n^T A_i e_n$, where $e_n \in \mathbf{R}^n$ is the vector whose entries are zero everywhere except the last entry, which is one.

  Define $\mathcal{Q}_0 = I$ and $\mathcal{Q}_i = \mathcal{Q}_{i-1} Q_i$ for $i = 1, 2, \cdots$. It follows that $A_i = \mathcal{Q}_i^T A \mathcal{Q}_i$. Further let q_i and $\widehat{q}_i$ be the last columns of Q_i and $\mathcal{Q}_i$, respectively. It is easy to see that $\widehat{q}_i = \mathcal{Q}_i q_i$. With this notation,
  $$\sigma_i = e_n^T \mathcal{Q}_i^T A \mathcal{Q}_i e_n = \widehat{q}_i^T A \widehat{q}_i.$$
  Let γ_i be the (n, n) entry of R_i. From the QR iteration, we have
  $$q_i^T \left(A_i - \sigma_i I\right) = q_i^T \left(Q_i R_i\right) = e_n^T R_i = \gamma_i e_n^T,$$
  which implies that
  $$q_i = \gamma_i \left(A_i - \sigma_i I\right)^{-1} e_n = \gamma_i \mathcal{Q}_i^T \left(A - \sigma_i I\right)^{-1} \mathcal{Q}_i e_n.$$
  Hence
  $$\mathcal{Q}_i q_i = \gamma_i \left(A - \sigma_i I\right)^{-1} \mathcal{Q}_i e_n.$$
  In other words,
  $$\widehat{q}_{i+1} = \gamma_i \left(A - \sigma_i I\right)^{-1} \widehat{q}_i,$$
  where $\sigma_i = \widehat{q}_i^T A \widehat{q}_i$ and where γ_i is a normalizing factor. This is clearly the Rayleigh quotient iteration.

- Let $A = Q\Lambda Q^*$ be the eigendecomposition of A, with $Q = [q_1, \cdots, q_n]$, and let the initial vector $x_0 = q_1 + q_2$. Show that RQI fails to converge in exact arithmetic. Run `rayleigh.m` with this initial vector to see what it does in finite precision.

  Solution: Let λ_1 and λ_2 be the eigenvalues corresponding to q_1 and q_2. We assume $\lambda_1 < \lambda_2$. It follows that $\rho_0 \stackrel{\text{def}}{=} \rho(x_0, A) = (\lambda_1 + \lambda_2)/2$. It follows that

  $$y_0 = (A - \rho_0 I)^{-1} x_0/\|x_0\|_2 = \left(\frac{\lambda_1 - \lambda_2}{2}\right)^{-1} \frac{1}{\sqrt{2}} (q_1 - q_2).$$

  Hence

  $$x_1 = y_0/\|y_0\|_2 = \frac{1}{\sqrt{2}} (q_2 - q_1).$$

  With one more step of RQI, we would get

  $$x_2 = \frac{1}{\sqrt{2}} (q_2 + q_1) = x_0.$$

  Hence RQI stalls.

- Let $B \in \mathbf{R}^{n\times n}$ be an upper bidiagonal matrix. Find explicit formulas for its inverse.

- Generate upper bidiagonal matrices of various dimensions, and run matlab code `BiSVD.m` (available on class website) to compute their smallest singular values. You should try different scalings on the diagonal entries so the smallest singular values can be really tiny ($10^{-100} - 10^{-50}$, for example).

  To check that these are indeed very accurate singular values, we use the formula

  $$1/\sigma_{\max}\left(B^{-1}\right) = \sigma_{\min}(B). \tag{1}$$

  The matlab `svd` function is backward stable. We generate B^{-1} explicitly using the explicit formulas. This way the largest singular value of B^{-1} is computed to full machine precision. Compare $1/\sigma_{\max}(B^{-1})$ with the singular values computed using `BiSVD.m` to show that `BiSVD.m` is highly accurate even for tiny singular values.

  Solution: Let $B = D + H$, where D is a non-singular diagonal matrix and H is non-zero only on its first super-diagonal. Define $\widehat{H} = D^{-1}H$, which has the same non-zero pattern as H. Since $\widehat{H}^n = 0$, it follows that

  $$B^{-1} = \left(I + \widehat{H}\right)^{-1} D^{-1} = \sum_{k=0}^{n-1} (-1)^k \widehat{H}^k D^{-1}.$$

  Let $D = \mathbf{diag}(\alpha_1, \cdots, \alpha_n)$ and

  $$\widehat{H} = \begin{pmatrix} 0 & \beta_1 & & & \\ & 0 & \beta_2 & & \\ & & \ddots & \ddots & \\ & & & & \beta_{n-1} \\ & & & & 0 \end{pmatrix}.$$

Then $\widehat{H}^k$ is a matrix that is non-zero only on its k-th super-diagonal, whose entries are

$$\prod_{j=i}^{i+k-1} \beta_j, \quad i = 1, \cdots, n-k.$$

- **Problem 5.28 Solution:** Let λ be an eigenvalue with eigenvector $x \neq 0$. Hence $Ax = \lambda x$ and $x^* A x = \lambda x^* x$. Taking complex conjugate of the last equation, we have

  $$\bar{\lambda} x^* x = x^* A^* x = -x^* A x = -\lambda x^* x,$$

  which leads to $\bar{\lambda} = -\lambda$. Hence λ is pure imaginary. Furthermore, this implies that the real part of every eigenvalue of $I - A$ must be 1. Hence $I - A$ does not have a zero eigenvalue, which means $I - A$ must be non-singular. Since

  $$C^* C = (I - A^*)^{-1} (I + A^*) (I - A)^{-1} (I + A) = (I + A)^{-1} (I - A) (I - A)^{-1} (I + A) = I,$$

  C is unitary.