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## Math221: Matrix Computations

### Homework #9 Solutions

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- Problems 5.1, 5.2, 5.4, 5.5, 5.6, 5.7, 5.18.
- **Problem 5.4 Solution:** We will only consider the second bullet. The first is a special case of the second with  $n = m + 1$ . Let  $\mathbf{R}^j$  be any  $j$  dimensional subspace of  $\mathcal{R}^m$ , let

$$\widehat{\mathbf{R}}^j = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{0}_{n-m} \end{pmatrix}, \quad \text{where } \mathbf{x} \in \mathbf{R}^j. \right\}$$

In other words,  $\widehat{\mathbf{R}}^j$  is the set of vectors obtained by padding zeros to the vectors in  $\mathbf{R}^j$ . It is easy to show that  $\widehat{\mathbf{R}}^j$  is a  $j$  dimensional subspace of  $\mathcal{R}^n$ . For any non-zero vector  $u \in \mathbf{R}^j$ , let  $\widehat{u}$  be the corresponding vector in  $\widehat{\mathbf{R}}^j$ . We can easily verify that

$$\frac{u^T H u}{u^T u} = \frac{\widehat{u}^T A \widehat{u}}{\widehat{u}^T \widehat{u}}.$$

In other words,  $\rho(u, H) = \rho(\widehat{u}, A)$ . Let  $\widetilde{\mathbf{R}}^j$  be any  $j$  dimensional subspace of  $\mathcal{R}^n$ . Since

$$\min_{0 \neq u \in \mathbf{R}^j} \rho(u, H) = \min_{0 \neq \widehat{u} \in \widehat{\mathbf{R}}^j} \rho(\widehat{u}, A) \leq \max_{\widetilde{\mathbf{R}}^j} \min_{0 \neq \widetilde{u} \in \widetilde{\mathbf{R}}^j} \rho(\widetilde{u}, A) = \alpha_j$$

by the Courant-Fischer minimax theorem, it follows by the same theorem that

$$\theta_j = \max_{\mathbf{R}^j} \min_{0 \neq u \in \mathbf{R}^j} \rho(u, H) \leq \alpha_j.$$

Similarly, let  $\mathbf{S}^j$  be a  $j$  dimensional subspace of  $\mathcal{R}^m$ , let  $\widehat{\mathbf{S}}^j$  be the set of vectors obtained by padding zeros to the vectors in  $\mathbf{S}^j$ . It is again easy to show that  $\widehat{\mathbf{S}}^j$  is a  $j$  dimensional subspace of  $\mathcal{R}^n$ . For any non-zero vector  $u \in \mathbf{S}^j$ , again let  $\widehat{u}$  be the corresponding vector in  $\widehat{\mathbf{S}}^j$ . We have

$$\max_{0 \neq u \in \mathbf{S}^{m-j+1}} \rho(u, H) = \max_{0 \neq \widehat{u} \in \widehat{\mathbf{S}}^{m-j+1}} \rho(\widehat{u}, A) \geq \min_{\widetilde{\mathbf{S}}^{m-j+1}} \max_{0 \neq \widetilde{u} \in \widetilde{\mathbf{S}}^{m-j+1}} \rho(\widetilde{u}, A) = \alpha_{j+n-m}$$

by the Courant-Fischer minimax theorem, it follows by the same theorem that

$$\theta_j = \min_{\mathbf{S}^{m-j+1}} \max_{0 \neq u \in \mathbf{S}^{m-j+1}} \rho(u, H) \geq \alpha_{j+n-m}.$$

- **Problem 5.5 Solution:** First of all, for any non-zero vector  $u \in \mathbf{R}^n$ , we have

$$\theta_n \leq \frac{u^T H u}{u^T u} \leq \theta_1.$$

It follows that

$$\frac{u^T A u}{u^T u} + \theta_n \leq \frac{u^T (A + H) u}{u^T u} \leq \frac{u^T A u}{u^T u} + \theta_1$$

for any non-zero vector  $u \in \mathbf{R}^n$ . By the Courant-Fischer minimax theorem, we have

$$\lambda_j = \min_{\mathbf{S}^{n-j+1}} \max_{0 \neq u \in \mathbf{S}^{n-j+1}} \frac{u^T (A + H) u}{u^T u} \leq \min_{\mathbf{S}^{n-j+1}} \max_{0 \neq u \in \mathbf{S}^{n-j+1}} \left( \frac{u^T A u}{u^T u} + \theta_1 \right) = \alpha_j + \theta_1.$$

Similarly,

$$\lambda_j = \min_{\mathbf{S}^{n-j+1}} \max_{0 \neq u \in \mathbf{S}^{n-j+1}} \frac{u^T (A + H) u}{u^T u} \geq \min_{\mathbf{S}^{n-j+1}} \max_{0 \neq u \in \mathbf{S}^{n-j+1}} \left( \frac{u^T A u}{u^T u} + \theta_n \right) = \alpha_j + \theta_n.$$

- **Problem 5.6 Solution:** Let

$$\mathcal{A} = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} = \begin{pmatrix} 0 & A_1 & A_2 \\ A_1^T & 0 & 0 \\ A_2^T & 0 & 0 \end{pmatrix}$$

and let

$$\mathcal{A}_1 = \begin{pmatrix} 0 & A_1 \\ A_1^T & 0 \end{pmatrix}.$$

Then  $\tau_j$  and  $\sigma_j$  are the  $j$ -th largest eigenvalues of  $\mathcal{A}_1$  and  $\mathcal{A}$ , respectively. Since  $\mathcal{A}_1 \in \mathbf{R}^{(n+m) \times (n+m)}$  is a leading principle submatrix of  $\mathcal{A} \in \mathbf{R}^{(2n) \times (2n)}$ , it follows from Problem 5.4 that

$$\sigma_j \geq \tau_j \geq \sigma_{j+n-m}.$$

- **Problem 5.7 Solution:** We only consider the case where  $d \neq 0$ . Let  $q_d = d/\|d\|_2$ . Since  $d$  is orthogonal to  $q$ , there must exist a matrix  $\hat{Q}$  such that  $Q \stackrel{\text{def}}{=} \begin{pmatrix} q & q_d & \hat{Q} \end{pmatrix} \in \mathbf{R}^{n \times n}$  is an orthogonal matrix. It follows that

$$(q + d) q^T - I = Q \begin{pmatrix} 1 \\ \|d\|_2 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} Q^T - Q Q^T = Q \begin{pmatrix} 0 & 0 & 0 \\ \|d\|_2 & -1 & \\ 0 & 0 & -I \end{pmatrix} Q^T.$$

Hence

$$\|(q + d) q^T - I\|_2 = \left\| \begin{pmatrix} 0 & 0 & 0 \\ \|d\|_2 & -1 & \\ 0 & 0 & -I \end{pmatrix} \right\|_2 = \sqrt{1 + \|d\|_2^2} = \|q + d\|_2.$$