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Math221: Matrix Computations
Homework #8 Solutions

- **Problem 4.6:**

1. Let $A = Q_A A' Q_A^*$ and $B = Q_B B' Q_B^*$ be the Schur decompositions of A and B, respectively. Both A' and B' are upper triangular and Q_A and Q_B are unitary. It follows that
$$Q_A A' Q_A^* X - X Q_B B' Q_B^* = C,$$
and that
$$A'Y - YB' = C', \tag{1}$$
where $Y = Q_A^* X Q_B$ and $C' = Q_A^* C Q_B$. Once we have solved for Y, we can recover X by computing $X = Q_A Y Q_B^*$.

2. Partition
$$A' = \begin{pmatrix} \tilde{A} & a \\ & \alpha \end{pmatrix}, \quad B' = \begin{pmatrix} \tilde{B} & b \\ & \beta \end{pmatrix} \quad \text{and} \quad C' = \begin{pmatrix} \tilde{C} & c_1 \\ c_2^* & \gamma \end{pmatrix}$$
and
$$Y = \begin{pmatrix} \tilde{Y} & y_1 \\ y_2^* & \delta \end{pmatrix}.$$
It follows from (1) that
$$\begin{aligned} \alpha y_2^* - y_2^* \tilde{B} &= c_2^* & (2) \\ (\alpha - \beta)\,\delta - y_2^* b &= \gamma & (3) \\ \tilde{A} y_1 + \delta a - \tilde{Y} b - \beta y_1 &= c_1 & (4) \\ \tilde{A}\tilde{Y} + a y_2^* - \tilde{Y}\tilde{B} &= \tilde{C}. & (5) \end{aligned}$$
Equation (2) implies
$$y_2^* \left(\alpha I - \tilde{B}\right) = c_2^*,$$
which has a unique solution for any c_2 when all eigenvalues of $\tilde{B}$ differ from α. Having computed y_2^*, we can compute δ from equation (3) as
$$(\alpha - \beta)\,\delta = y_2^* b + \gamma.$$

This equation has a unique solution when α differs from β. Overall, we can solve both y_2 and δ uniquely as long as α is not an eigenvalue of B'.

From equation (5), we can now proceed to recursively compute the lower triangular part of Y from the following equation:

$$\tilde{A}\tilde{Y} - \tilde{Y}\tilde{B} = \tilde{C} - a y_2^*.$$

As long as A' and B' do not share eigenvalues, we can proceed to uniquely determine the lower triangular part of Y.

Having done so, we can now further determine the strictly upper triangular part of Y, one column at a time, from left to right. With an induction argument, assuming we have determined all upper triangular components of $\tilde{Y}$, so that all components of $\tilde{Y}$ have been determined, we can now proceed to determine y_1 from equation (4) as

$$\left(\tilde{A} - \beta I\right) y_1 = c_1 - \delta a + \tilde{Y} b.$$

This equation has a unique solution when A' and B' do not share a common eigenvalue.

To recap, we solve the lower triangular part of Y from right to left, one column at a time, including the diagonals. Once this is done, we solve for the strictly upper triangular part of Y from left to right, one column at a time. (Alternatively, we can also solve for Y from top left to bottom right, one column and one row at a time.)

- **Problem 4.7:** Since $S^{-1} = \begin{pmatrix} I & -R \\ & I \end{pmatrix}$, we have

$$S^{-1}TS = \begin{pmatrix} I & -R \\ & I \end{pmatrix} \begin{pmatrix} A & C \\ & B \end{pmatrix} \begin{pmatrix} I & R \\ & I \end{pmatrix} = \begin{pmatrix} A & C + AR - RB \\ & B \end{pmatrix}.$$

Hence we can choose R such that

$$R\ B - A\ R = C.$$

In order for this equation to indeed have a solution, we require that A and B have no common eigenvalues.

- **Problem 4.8:** Let $X = \begin{pmatrix} I & -A \\ & I \end{pmatrix}$. Then $X^{-1} = \begin{pmatrix} I & A \\ & I \end{pmatrix}$ and

$$X \begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix} X^{-1} = \begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}.$$

Since eigenvalues of $\begin{pmatrix} AB & A \\ B & 0 \end{pmatrix}$ are those of AB and 0's, and eigenvalues of $\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$ are those of BA and 0, and since these matrices are similar, it follows that both AB and BA have the same set of non-zero eigenvalues.

- **Problem 4.10:**

1. $H = (A + A^*)/2$ is Hermitian, whereas $S = (A - A^*)/2$ is skew-Hermitian, with $A = H + S$. This decomposition is unique.
2. Let $A = QUQ^*$ be the Schur form for A, where Q is unitary and U upper triangular. Then the eigenvalues of A are simply the diagonal entries of U. Let $U = \widehat{H} + \widehat{S}$, where $\widehat{H}$ Hermitian and $\widehat{S}$ skew-Hermitian. It follows that the diagonal entries on the main diagonal of $\widehat{H}$ are the real parts of the eigenvalues of A. Consequently,

$$\sum_i |\mathcal{R}(\lambda_i)|^2 \leq \|\widehat{H}\|_F^2.$$

Since

$$\widehat{H} = (U + U^*)/2 = (Q^*AQ + (Q^*AQ)^*)/2 = Q^*HQ,$$

it follows that

$$\|\widehat{H}\|_F = \|H\|_F^2.$$

Hence

$$\sum_i |\mathcal{R}(\lambda_i)|^2 \leq \|H\|_F^2.$$

3. Continue the arguments above, since and the diagonal entries on the main diagonal of $\widehat{S}$ are the imaginary parts of the eigenvalues of A, and since $\|\widehat{S}\|_F = \|S\|_F$, we have

$$\sum_i |\mathcal{I}(\lambda_i)|^2 \leq \|S\|_F^2.$$

4. Again let $A = QUQ^*$ be the Schur form for A. Then we have to prove that A is normal if and only U is. So we have to prove that A is normal if and only if

$$\sum_i |\lambda_i|^2 = \|U\|_2^2.$$

On the other hand, since λ_i's are the eigenvalues of A, they must be on the diagonal of U. The last equation therefore implies that all off-diagonal entries of U must vanish. Hence U is diagonal and hence $A = QUQ^*$ must be normal.