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Math221: Matrix Computations
Homework #7 Solutions

- Let $A \in \mathbf{R}^{n\times n}$ be non-singular. The QR factorization with column pivoting gives

$$A\Pi = QR,$$

where Π is a permutation. Let D be the diagonal of R and $U = D^{-1}R$, so that U is an upper triangular matrix with unit diagonal entries. This leads to

$$A\Pi = Q\,D\,U.$$

  – Show that $\|U\|_{\max} = 1$.

    Solution: By definition of column pivoting, we have that $|R_{i,i}| \geq \|R_{i:n,j}\|_2$ for all i and all $j > i$. This implies that all diagonal entries of U are 1 and no off-diagonal entry of U can be bigger than 1.

  – Show that $\|U^{-1}\|_{\max} \leq 2^{n-1}$.

    Solution: Let $W = U^{-1}$. We will use induction on n to show that the diagonal entries of W are all 1 and

$$|W_{i,j}| \leq 2^{j-i-1}, \quad \text{for all} \quad j > i. \tag{1}$$

    This is obviously true for $n = 1$ and $n = 2$. Assuming this is also true for $n-1 \geq 2$. Partition

$$U = \begin{pmatrix} \widehat{U} & u \\ & 1 \end{pmatrix}.$$

    Then

$$W = \begin{pmatrix} \widehat{W} & -\widehat{W}u \\ & 1 \end{pmatrix},$$

    where $\widehat{W} = \widehat{U}^{-1}$ is also upper triangular. By induction assumption, the diagonal entries of $\widehat{W}$ are all 1 and

$$|\widehat{W}_{i,j}| \leq 2^{j-i-1}, \quad \text{for all} \quad j > i.$$

With this assumption, we see that all diagonal entries of W are 1 and all off-diagonal entries of W satisfy equation (1) except those in its last column.
Since $\|u\|_{\max} \leq 1$, it follows that the j-th entry of $\widehat{W}u$ is bounded in absolute value by

$$1 + \sum_{i=j+1}^{n-1} 2^{i-j-1} = 2^{n-j-1}.$$

Hence equation (1) indeed holds for matrix dimension n.

– For different values of n and c, compute $\|U^{-1}\|_{\max}$ for the Kahan matrix (kahan.m at class website).

- **Problem 4.2:** For any square matrix A, let its Schur form be $A = QTQ^*$, where T is upper triangular and Q is unitary. Since

$$AA^* = Q\left(TT^*\right)Q^* \quad \text{and} \quad A^*A = Q\left(T^*T\right)Q^*,$$

it follows that A is normal if and only if T is normal.

- **Problem 4.3:** We have

$$Ax = \lambda x \quad \text{and} \quad y^*A = \mu y^*.$$

Hence

$$\lambda y^*x = y^*\left(\lambda x\right) = y^*Ax = \left(y^*A\right)x = \left(\mu y^*\right)x = \mu y^*x,$$

which implies $\left(\lambda - \mu\right)y^*x = 0$, or $y^*x = 0$.

- **Problem 4.4:**

1. $$f(A) = \sum_{i=-\infty}^{+\infty} a_i A^i = Q\left(\sum_{i=-\infty}^{+\infty} a_i T^i\right)Q^* = Qf(T)Q^*.$$

2. Since T is upper triangular, the (i,i) entry of T^j is $\left(T_{i,i}\right)^j$. Hence the (i,i) entry of $f(T)$ is $\sum_{j=-\infty}^{+\infty} a_j\left(T_{i,i}\right)^j = f(T_{i,i})$.

3. $$Tf(T) = T\left(\sum_{i=-\infty}^{+\infty} a_i T^i\right) = \left(\sum_{i=-\infty}^{+\infty} a_i T^i\right)T = f(T)T.$$

4. Partition

$$T = \begin{pmatrix} \widehat{T} & t \\ & \tau \end{pmatrix}, \quad \text{and} \quad f(T) = \begin{pmatrix} f\left(\widehat{T}\right) & F \\ & f(\tau) \end{pmatrix}.$$

The earlier equation becomes

$$\begin{pmatrix} \widehat{T} & t \\ & \tau \end{pmatrix}\begin{pmatrix} f\left(\widehat{T}\right) & F \\ & f(\tau) \end{pmatrix} - \begin{pmatrix} f\left(\widehat{T}\right) & F \\ & f(\tau) \end{pmatrix}\begin{pmatrix} \widehat{T} & t \\ & \tau \end{pmatrix} = 0,$$

which expands to

$$\begin{pmatrix} \widehat{T} f\left(\widehat{T}\right) & \widehat{T} F + f(\tau) t \\ & \tau f(\tau) \end{pmatrix} - \begin{pmatrix} f\left(\widehat{T}\right) \widehat{T} & f\left(\widehat{T}\right) t + \tau F \\ & f(\tau)\tau \end{pmatrix}.$$

The $(1,2)$ block of this equation yields

$$\left(\widehat{T} - \tau I\right) F = \left(f\left(\widehat{T}\right) - f(\tau) I\right) t, \tag{2}$$

which can be used to solve for F given $f\left(\widehat{T}\right)$.

This suggests an algorithm to solve for $f(T)$ as follows: We first compute all the diagonals of $f(T)$. We then apply equation (2) to every 2×2 block submatrix along the main diagonal to solve for all the entries in the first superdiagonal of $f(T)$. Similarly, we then apply equation (2) to every 3×3 block submatrix along the main diagonal to solve for all the entries in the second superdiagonal of $f(T)$, and so on. Mathematically, this procedure will always work since the coefficient matrix $\widehat{T} - \tau I$ is always non-singular given that eigenvalues of A are distinct.