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Math221: Matrix Computations
Homework #2 Solutions

- **2.3:** We need relevent vectors in equation (2.1) to have the same direction in order for (2.3) to be an equation. Hence we let

$$A^{-1}\delta b = \alpha \delta x, \quad A^{-1}\delta A \hat{x} = -\beta \delta x,$$

where α and β are scalars to be determined. This leads to

$$\delta b = \alpha A \delta x \quad \text{and} \quad \delta A \hat{x} = -\beta A \delta x.$$

With the trick in the proof of Theorem 2.1, we conclude that the minimun 2-norm solution for δA from the above equation is

$$\delta A = -\frac{\beta A \delta x \hat{x}^T}{\hat{x}^T \hat{x}}. \tag{1}$$

With these choices, the right hand side of inequality (2.3) becomes $(|\alpha| + |\beta|) \|A^{-1}\|_2 \|A\delta x\|_2$, whereas the right hand side of (2.1) becomes $(\alpha + \beta)\, \delta x$. It is obvious that for (2.3) to be an equation, we must first set $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta = 1$. With these conditions, the equation

$$(A + \delta A)\, \hat{x} = b + \delta b \tag{2}$$

holds for any $\hat{x} = x + \delta x$ as long as $\hat{x} \neq 0$. Hence we can choose $\hat{x}$ in anyway we wish, as long as $\|\delta x\|_2$ is sufficiently small.

With the choice of α and β, the right hand side of inequality (2.3) has become $\|A^{-1}\|_2 \|A\delta x\|_2$. Let

$$A\delta x = \gamma u, \quad \text{or} \quad \delta x = \gamma A^{-1} u,$$

where $\gamma \geq 0$ is a scalar to be determined, and u is a unit vector such that $\|A^{-1}\|_2 = \|A^{-1}u\|_2$. For any δx in the above form, equation (2) is satisfied and (2.2) becomes an exact equation. In the following, we have to further specify γ, with given values of $\|\delta A\|_2$ and $\|\delta b\|_2$, which are assumed to be sufficiently small.

In fact, with our choice of δx, we have

$$\delta b = \alpha\gamma u \quad \text{and} \quad \delta A = -\frac{\beta\gamma u \hat{x}^T}{\hat{x}^T\hat{x}}.$$

Taking norms on both equations, we have

$$\alpha\gamma = \|\delta b\|_2 \quad \text{and} \quad \beta\gamma = \|\delta A\|_2\|\hat{x}\|_2.$$

Together with $\alpha + \beta = 1$, we have

$$\alpha = \frac{\|\delta b\|_2}{\gamma}, \quad \beta = \frac{\|\delta A\|_2\|\hat{x}\|_2}{\gamma},$$

where

$$\gamma = \|\delta b\|_2 + \|\delta A\|_2\|\hat{x}\|_2 = \|\delta b\|_2 + \|\delta A\|_2\|x + \gamma A^{-1}u\|_2, \tag{3}$$

which is a non-linear equation in γ. For the above choices to make sense, we now must show that this equation does have a solution.

One way to show that this equation has indeed a solution is to use fixed-point iteration. Define $\gamma_0 = \|\delta b\|_2 + \|\delta A\|_2\|x\|_2$ and

$$\gamma_{k+1} = \|\delta b\|_2 + \|\delta A\|_2\|x + \gamma_k A^{-1}u\|_2, \quad k = 0, 1, 2, \cdots.$$

One can show that for all $k > 1$,

$$\begin{aligned} 0 \leq \gamma_k &\leq \frac{\|\delta b\|_2 + \|\delta A\|_2\|x\|_2}{1 - \|\delta A\|_2\|A^{-1}\|_2}, \\ |\gamma_{k+1} - \gamma_k| &\leq |\gamma_k - \gamma_{k-1}|\|\delta A\|_2\|A^{-1}\|_2. \end{aligned}$$

Hence the γ sequence must be convergent provided that $\|\delta A\|_2\|A^{-1}\|_2 < 1$, which is true since $\|\delta A\|_2$ is assumed to be sufficiently small. The limit of this sequence is the solution of the above non-linear equation.