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# Math170: Mathematical Methods for Optimization Final

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**This is a closed book, closed notes exam. You need to justify every one of your answers. Completely correct answers given without justification will receive little credit. Do as much as you can. Partial solutions will get partial credit. Look over the whole exam to find problems that you can do quickly. You need not simplify your answers unless you are specifically asked to do so.**

| Problem | Maximum Score | Your Score |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 20 | |
| Total | 100 | |

Your Name: ______________________________

Your SID: ______________________________

1. For any given vector $v = (v_1, \cdots, v_n)^T$, its 1-norm is defined as $\|v\|_1 = |v_1| + \cdots |v_n|$.

   (a) Show that the 1-norm minimization problem

   $$\min_x \|Ax - b\|_1$$

   is equivalent to the LP

   $$\begin{aligned} &\min_{u,x} e^T u, \\ &s.t. -u \leq Ax - b \leq u, \end{aligned}$$

   where $e = (1, \cdots, 1)^T$.

   (b) Find the dual of the above LP.

2. Let $C \in \mathbf{R}^{n \times n}$ be a symmetric positive semidefinite matrix, and let $A \in \mathbf{R}^{m \times n}$, $p \in \mathbf{R}^n$, $b \in \mathbf{R}^m$. The following is a QP in canonical form:

$$\begin{aligned} &\min \frac{1}{2} x^T C x + p^T x, \\ &s.t. \quad Ax = b, \\ &x \geq 0. \end{aligned}$$

The first order optimality conditions for this QP are known as follows:

$$\begin{aligned} Cx + A^T u - v &= -p, \\ Ax &= b, \\ x \cdot v &= 0, \end{aligned}$$

where $v \in \mathbf{R}^n$ is a non-negative vector; and $u \in \mathbf{R}^m$. The notation $x \cdot v$ denotes the component-wise product of $x$ and $v$.

Now consider instead the following QP:

$$\begin{aligned} &\min \frac{1}{2} x^T C x + p^T x, \\ &s.t. \quad 0 \leq x \leq u, \end{aligned}$$

where $u > 0$ is the upper bound. Derive the first order optimality conditions for this QP.

3. Let $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$. The Farkas Lemma asserts that either

   (a) The equations $Ax = b$ have a solution $x \geq 0$, or

   (b) $A^T y \geq 0$ has a solution $y$ such that $b^T y < 0$,

   but not both. Use this lemma to show that either

   (a) The equations $Ax = b$ have a solution $0 \leq x \leq u$, or

   (b) $A^T y_1 + y_2 \geq 0$ has a solution $(y_1^T, y_2^T)$ with $y_2 \geq 0$ such that $b^T y_1 + u^T y_2 < 0$,

   but not both.

4. Consider the linear program

$$
\begin{aligned}
&\min x_1 + 2x_2, \\
&s.t. x_1 + x_2 \leq 1, \\
&x_1 - x_2 \leq 0, \\
&x_1, x_2 \geq 0.
\end{aligned}
$$

   (a) Draw the region of feasibility.

   (b) Solve the LP and verify the optimality of your optimal solution.

5. Consider the network flow problem for $N = s, a, b, s'$, with the capacity $\mathbf{k}$ where

$$\mathbf{k}(s,a) = 1, \quad \mathbf{k}(s,b) = 3, \quad \mathbf{k}(a,b) = 0, \quad \mathbf{k}(b,a) = 2, \quad \mathbf{k}(a,s') = 4, \quad \mathbf{k}(b,s') = 1.$$

Find the maximum flow for this problem.