Homework 3 Math 170
Reader: Michael Pejic mpejic@math.berkeley.edu

Problem set: *Franklin* Section 1.5, Problems 1,2,4,5,10; Section 1.4, Problems 1,2,3,5

Problems checked: *Franklin* Section 1.3, Problems 1,2; Section 1.4, Problems 1,2,3

Grading scheme:

X	for "complete": significant effort demonstrated
O	for "fail": lack of demonstration of significant effort

Problems graded: *Franklin Franklin* Section 1.5, Problems 4,5,10; Section 1.6, Problem 5

Grading scheme:

3	for "excellent":	Necessary steps are all shown and well explained. Solution is correct.
2	for "fair":	Necessary steps are all shown. There are minor gaps in explanantion and/or minor errors in solution.
1	for "poor":	Necessary steps are lacking. There are major gaps in explanantion and/or major errors in solution.
0	for "fail":	Significant effort is not demonstrated.

Sample solutions:

***Franklin* Section 1.5, Problem 4** The three basic solutions are given by $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, the last two of which are feasible. Writing the problem as a Phase I problem, it becomes

$$\begin{bmatrix} -1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{x} \geq 0, \mathbf{z} \geq 0, \text{minimize } z_1 + z_2$$

Starting with the basic feasible solution $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ with $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = M^{-1}$, a Phase II calculation involving the second column of the matrix gives $\begin{bmatrix} t_4 \\ t_5 \end{bmatrix} = M^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, p = 5, \lambda^* = 1$, leading to a a new feasible solution $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ with $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = M^{-1}$. Another Phase II calculation involving the third column of the matrix gives $\begin{bmatrix} t_2 \\ t_4 \end{bmatrix} = M^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, p = 4, \lambda^* = 1$, leading to a new feasible solution $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, which corresponds to the feasible solution $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ in the first step.

Alternatively, starting with the same basic feasible solution, if instead a Phase II calculation involving the first column of the matrix were done, it would give $\begin{bmatrix} t_4 \\ t_5 \end{bmatrix} = M^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, p = 5, \lambda^* = 1$, leading to a new

feasible solution $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ with $M = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \Leftrightarrow M^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Another Phase II calculation involving the third column of the matrix gives $\begin{bmatrix} t_1 \\ t_4 \end{bmatrix} = M^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, p = 4, \lambda^* = 2$, leading to a new feasible solution $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$, which corresponds to the feasible solution $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ in the first step.

***Franklin* Section 1.5, Problem 5** Clearly the feasible basic solution $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ minimizes $x_1 + x_2 + x_3$. To confirm this result using a Phase II calculation, start with this feasible solution and perform a Phase II calculation involving the first column of the matrix. $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = M^{-1}$, so $\begin{bmatrix} t_2 \\ t_3 \end{bmatrix} M^{-1}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Since $c_1 = c_2 = c_3 = 1$, so $t_2c_2 + t_3c_3 - c_1 = -1 < 0$, we are in Case 1 and our basic feasible solution is optimal.

***Franklin* Section 1.5, Problem 10** The dual program is

$$\text{maximize } \mathbf{y}^{\mathrm{T}}\begin{bmatrix} 3+\varepsilon \\ 3 \end{bmatrix} \text{ with } \mathbf{y}^{\mathrm{T}}\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \leq \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

The equilibrium constraints are that

$$\mathbf{y}^{\mathrm{T}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0 \text{ if } x_1 > 0, \mathbf{y}^{\mathrm{T}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \text{ if } x_2 > 0, \mathbf{y}^{\mathrm{T}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0 \text{ if } x_3 > 0$$

One possible basic feasible solution is to take $x_2 = 0$; then

$$\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3+\varepsilon \\ 3 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3+\varepsilon \\ 3 \end{bmatrix} = \begin{bmatrix} 1-\frac{1}{3}\varepsilon \\ 1+\frac{2}{3}\varepsilon \end{bmatrix}$$

which is feasible for ε sufficiently small (in the interval $\left[-\frac{3}{2}, 3\right]$). This is optimal since then the equilibrium constraints are satisfied by the feasible solution $\mathbf{y} = 0$. Alternatively, a Phase II calculation could be made involving the second column of the matrix, which would lead to the same result. In the limit as $\varepsilon \to 0$, the optimal solution goes to $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

***Franklin* Section 1.6, Problem 5** The closed disc is distance 4 units from the origin, so $\delta = 4$. The closest point to the origin, $\mathbf{x}_0$, is $\left(-\frac{12}{5}, \frac{16}{5}\right)$. The unit vector in the direction of the closest point, $\mathbf{u}$, is $\left(-\frac{3}{5}, \frac{4}{5}\right)$. The midpoint of the line segment from the origin to $\mathbf{x}_0$, $\mathbf{m}$, is $\left(-\frac{6}{5}, \frac{8}{5}\right)$. The equation of a separating plane is then $\mathbf{u}^{\mathrm{T}}(\mathbf{x} - \mathbf{m}) = 0$, or

$$\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \end{bmatrix}\left(\mathbf{x} - \begin{bmatrix} -\frac{6}{5} \\ \frac{8}{5} \end{bmatrix}\right) = 0$$