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Math170 Homework Set 5, Spring 2007

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1. **Problem 1:** Let $x = (x_1, x_2)^T$ and $x' = (x'_1, x'_2)^T$ be in the half plane:

   $$(3 \quad -5)x < 7, \quad (3 \quad -5)x' < 7.$$

   Let $0 \leq \theta \leq 1$. Then

   $$(3 \quad -5)(\theta x + (1-\theta)x') < 7.$$

   Hence the half plane is convex.

   But it is not closed, since the sequence

   $$\begin{pmatrix} 7/3 - 1/n \\ 0 \end{pmatrix}$$

   for $n = 1, 2, \cdots, ...$ is in the half plane, but its limit

   $$\begin{pmatrix} 7/3 \\ 0 \end{pmatrix}$$

   is not.

2. **Problem 2:** Let $x = (x_1, x_2)^T$ and $x' = (x'_1, x'_2)^T$ be in the half plane:

   $$(3 \quad -5)x \leq 7, \quad (3 \quad -5)x' \leq 7.$$

   Let $0 \leq \theta \leq 1$. Then

   $$(3 \quad -5)(\theta x + (1-\theta)x') \leq 7.$$

   Hence the half plane is convex.

   And it is closed. Let ${x^n}_{n=1}^{\infty}$ be a sequence in the half plane, that converges to x^*, then

   $$(3 \quad -5)x^n \leq 7$$

   for all n. Taking limit,

   $$(3 \quad -5)x^* \leq 7.$$

   Hence x^* is also in the half plane.

3. **Problem 3:** Let ${x^n}_{n=1}^{\infty}$ be a sequence in the annulus, that converges to x^*, then
$$1 \le |x^n| \le 2$$
for all n. Taking limit,
$$1 \le |x^*| \le 2$$
Hence x^* is also in the annulus, hence the closedness.

   Let x be in the annulus. Hence
$$1 \le |x| \le 2 \quad \text{and} \quad 1 \le |-x| \le 2.$$
In other words, both x and $-x$ are in the set. Let $\theta = 1/2$. Then $\theta x + (1-\theta)(-x) = 0$, which is not in the annulus, which means it can not be convex.

4. **Problem 4:** Let
$$C =< x^1, \cdots, x^p >$$
be the polytope. Let
$$x = \theta_1 x^1 + \cdots \theta_p x^p$$
and
$$x' = \theta_1^{'} x^1 + \cdots \theta_p^{'} x^p$$
be in C, where
$$\theta_1 + \cdots + \theta_p = 1, \theta_1 \ge 0, \cdots, \theta_p \ge 0.$$
and
$$\theta_1^{'} + \cdots + \theta_p^{'} = 1, \theta_1^{'} \ge 0, \cdots, \theta_p^{'} \ge 0.$$
Let $0 \le \theta \le 1$. Then
$$x'' = \theta x + (1-\theta)x' = (\theta\theta_1 + (1-\theta)\theta_1^{'})x^1 + \cdots + (\theta\theta_p + (1-\theta)\theta_p^{'})x^p.$$
The new θ coefficients are non-negative and add up to 1. Hence $x'' \in C$. This proves convexity.

   To prove closeness, we take a convergent sequence $\{y^n\}_{n=1}^{\infty}$ in C. Write
$$y^n = \theta_1^n x^1 + \cdots \theta_p^n x^p,$$
where $\theta_1^n + \cdots + \theta_p^n = 1$ and $\theta_1^n \ge 0, \cdots \theta_p^n \ge 0$.

   Since all the $\theta's$ are bounded between 0 and 1, we can pick a subsequence that converges. Let's say we have
$$\lim_{k \to \infty} \theta_j^{n_k} = \theta_j^*$$
for all $1 \le j \le p$. Then $\theta_1^* + \cdots + \theta_p^* = 1$ and $\theta_1^* \ge 0, \cdots \theta_p^* \ge 0$. Hence $\theta_1^* x^1 + \cdots + \theta_p^* x^p$ is also in C.

   Since $\{y^n\}_{n=1}^{\infty}$ converges, its subsequence $\{y^{n_k}\}_{k=1}^{\infty}$ converges as well and to the same limit, which is $\theta_1^* x^1 + \cdots + \theta_p^* x^p$. Hence C must be closed.

5. **Problem 5:** Much of this can be done with geometrical inspection.

   (i) $\delta = 4$.

   (ii) $x^0 = (-12/5, 16/5)^T$, $u = (-3/5, 4/5)^T$, and $m = (-6/5, 8/5)^T$.

   (iii) $(-3/5, 4/5) \cdot (x - (-6/5, 8/5)^T) = 0$.

6. **Problem 8:** Points $(1,0)^T$ and $(-1,0)^T$ are both in F, but their average $(0,0)^T$ is not. Hence F is not convex.

   Both $(1/2, 1/2)^T$ and $(-1/2, 1/2)^T$ are in F and are closest to $(0,0)^T$. Take the alternating sequence

   $$(1/2, 1/2)^T, (-1/2, 1/2)^T, (1/2, 1/2)^T, (-1/2, 1/2)^T, \cdots (1/2, 1/2)^T, (-1/2, 1/2)^T, ...$$

   They are all in F with the same value $|x^k|$. Yet they diverge.

7. **Problem 9:** Any sequence in **Problem 8** must be bounded, and have a convergent subsequence. Since F is closed, such a subsequence has to converge to a point in F. Since the original sequence converges in length to δ, so does the subsequence.

8. **Problem 11:** C_1 and C_2 are disks. Points in C_1 have length larger than 4 by Pathagorean theorem, whereas points in C_2 have length at most 4, so the two sets do not intersect.

   The point $(-12/5, 16/5)^T$ is on the boundaries of both C_1 and C_2. Hence the distance between C_1 and C_2 is 0. Separating plane is

   $$(-12/5, 16/5)(x - (-12/5, 16/5)^T) = 0,$$

   which is not strict separation.

9. **Problem 12:** To show that C_1 is convex, let $x = (x_1, x_2)^T$ and $y = (y_1, y_2)^T$ be in C_1, then $x_1, y_1 > 0$ and $x_1 x_2 \geq 1, y_1 y_2 \geq 1$. Let $0 \leq \theta \leq 1$ and $z = \theta x + (1-\theta)y = (\theta x_1 + (1-\theta)y_1, \theta x_2 + (1-\theta)y_2)^T$. Then $(\theta x_1 + (1-\theta)y_1 > 0$ and

   $$((\theta x_1 + (1-\theta)y_1) * ((\theta x_2 + (1-\theta)y_2) = \theta^2 x_1 y_1 + (1-\theta)^2 x_2 y_2 + \theta(1-\theta)(x_1 y_2 + y_1 x_2).$$

   Since $x_1 y_1 \geq 1, x_2 y_2 \geq 1$, and since

   $$x_1 y_2 + y_1 x_2 \geq 2\sqrt{x_1 y_2 y_1 x_2} \geq 2$$

   by Cauchy-Schwartz inequality, the above equality becomes

   $$((\theta x_1 + (1-\theta)y_1) * ((\theta x_2 + (1-\theta)y_2) \geq \theta^2 + (1-\theta)^2 + 2\theta(1-\theta) = 1.$$

   Similarly we can prove C_2 is convex. Closedness is similar to **Problem 2**.

   The difference set is $C = x - y : x \in C_1, y \in C_2$ (see Page 49.) Theorem 3 shows C is convex. To show C is not closed, let $x^n = (1/n, n)^T \in C_1$ and $y^n = (-1/n, n)^T \in C_2$. Their difference is $x^n - y^n = (2/n, 0)^T \in C$. The limit of $x^n - y^n$ as n goes to infinity is $(0,0)^T$ is not in C, since C_1 and C_2 are disjoint.

   This example also shows $\delta = 0$. The separating plane is $x_1 = 0$, and the separation is strict.

10. **Problem 14:** Proof similar to that of **Problem 4**.

11. **Problem 15:** Let $x \in C$. Then we can write

$$x = \theta_1 x^1 + \cdots + \theta_p x^p$$

for $x^1, \cdots, x^p \in S$ and $\theta_1 + \cdots + \theta_p = 1$, $\theta_1 \geq 0, \cdots \theta_p \geq 0$. In other words,

$$\begin{pmatrix} x \\ 1 \end{pmatrix} = \theta_1 \begin{pmatrix} x^1 \\ 1 \end{pmatrix} + \cdots + \theta_p \begin{pmatrix} x^p \\ 1 \end{pmatrix}.$$

With arguments similar to those in the proof of the theorem on Page 29, we can conclude that we can rewrite $\begin{pmatrix} x \\ 1 \end{pmatrix}$ as a non-negative linear combination of at most $N+1$ of the $\begin{pmatrix} x^j \\ 1 \end{pmatrix}' s$.