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Math128B: Numerical Analysis Sample Midterm

This is a closed book, closed notes exam. You need to justify every one of your answers. Completely correct answers given without justification will receive little credit. Do as much as you can. Partial solutions will get partial credit. Look over the whole exam to find problems that you can do quickly.

Problem	Maximum Score	Your Score
1	30	
2	20	
3	20	
4	30	
Total	100	

Your Name: _____

Your SID: _____

1. (30 Points) Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$.

   (a) Compute the spectral radius of A.

   (b) For the linear system of equations $Ax = b$ where $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, define the Jacobi method.

   (c) Find out whether the Jacobi method converges for any initial guess x_0.

2. (20 Points) Let $\phi_1(x)$, $\cdots$, $\phi_n(x)$ be continuous functions that are linearly independent on $[-1, 1]$; and let $f(x)$ be a continuous function on $[-1, 1]$. Define set

$$\mathcal{S} = \{P(x), \text{where } P(x) = \alpha_1\phi_1(x) + \cdots \alpha_n\phi_n(x), \text{ and } \alpha_1, \cdots, \alpha_n \text{ are real constants.}\}$$

Consider the least squares problem

$$\min_{P(x)\in\mathcal{S}} \int_{-1}^{1} (f(x) - P(x))^2 \, dx.$$

Find the normal equation that defines the optimal $P(x)$.

3. (20 Points) The Hankel matrix is a matrix whose entries are a constant along each anti-diagonal. The following is an example of a 4×4 Hankel matrix:

$$H_4 = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 \\ h_2 & h_3 & h_4 & h_5 \\ h_3 & h_4 & h_5 & h_6 \\ h_4 & h_5 & h_6 & h_7 \end{pmatrix}.$$

Let $H \in \mathbf{R}^{n \times n}$ be a Hankel matrix and let $u \in \mathbf{R}^n$ be a vector. Sketch an algorithm to compute Hu in $O(n \log_2 n)$ operations, assuming that n is a power of 2.

Hint: Let $S_n \in \mathbf{R}^{n \times n}$ be the matrix that is 1 on the main anti-diagonal and 0 elsewhere. For example,

$$S_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Show that $S_n H$ is a Toeplitz matrix.

4. (30 Points)

5. Let

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

   Use the Gram-Schmidt process to find a set of 3 orthonormal vectors.