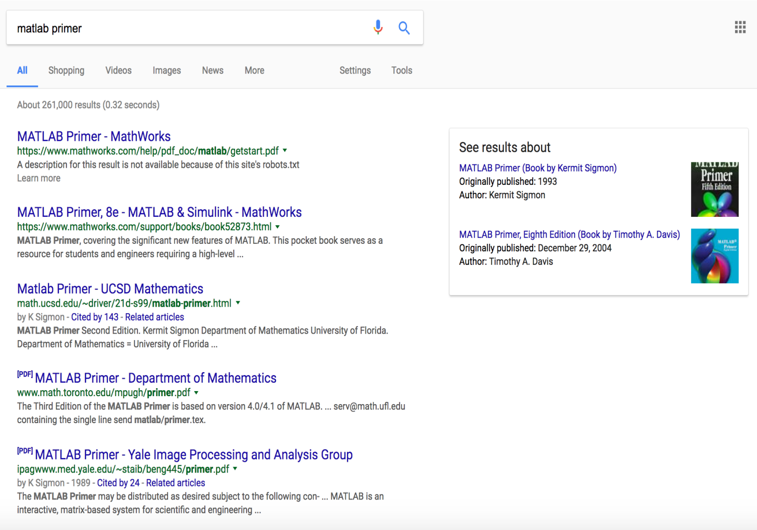
Matlab Primer



Math128A algorithms vs. General Purpose algorithms

For any vector $\mathbf{x} \in \mathbf{R}^n$, compute its norm

$$\|\mathbf{x}\|_2 = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{\frac{1}{2}} = \left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}}.$$

- **INPUT:** $n, x_1, \cdots, x_n$.
- **OUTPUT: Norm**.
- **Step 1**: Set **SUM** $= 0$.
- **Step 2**: For $k = 1, \cdots, n$ set **SUM** $=$ **SUM** $+ x_k * x_k$.
- **Step 3**: Set **Norm** $= \sqrt{\textbf{SUM}}$.
- **Step 4**: Output **Norm**.
  STOP.

Math128A algorithms vs. General Purpose algorithms

```
>> n=10;
>> x = (1:n)';
>> sum = 0;
>> for k=1:n
sum = sum + x(k)*x(k);
end
>> nrm = sqrt(sum);
>> disp([nrm, abs(nrm-sqrt(n*(n+1)*(2*n+1)/6))])
   1.9621e+01   0.0000e+00
```

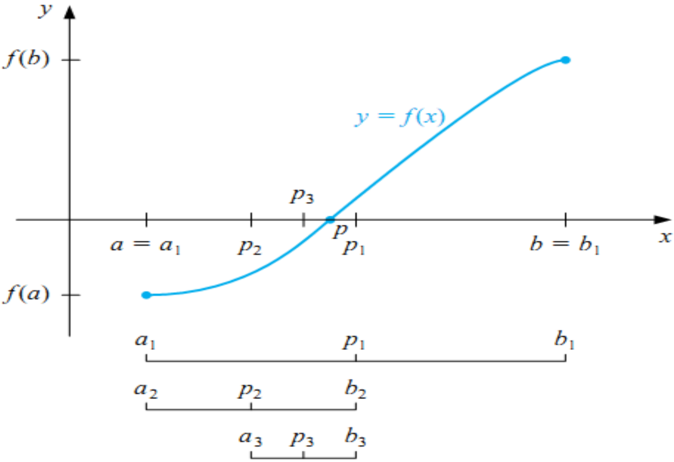
Math128A algorithms vs. General Purpose algorithms

```
>> n=10;
>> x = 1e200*(1:n)';
>> sum = 0;
>> for k=1:n
sum = sum + x(k)*x(k);
end
>> nrm = sqrt(sum);
>> disp([nrm, abs(norm(x)-1e200*sqrt(n*(n+1)*(2*n+1)/6))])
        Inf   0.0000e+00
```

Bisection Method

- **Given** continuous function $f(x)$ on the interval $[a, b]$ with $f(a) \cdot f(b) < 0$, there must be a root in (a, b).
- To find a root: set $[a_1, b_1] = [a, b]$.
- set $p_1 = \frac{a_1+b_1}{2}$ and compute $f(p_1)$.
  - if $f(p_1) = 0$, then quit with root p_1 (MUST BE VERY LUCKY.)
  - if $f(a_1) \cdot f(p_1) < 0$, then set $[a_2, b_2] = [a_1, p_1]$,
  - otherwise $(f(p_1) \cdot f(b_1) < 0)$ set $[a_2, b_2] = [p_1, b_1]$,
- repeat with $p_2 = \frac{a_2+b_2}{2}$.

Bisection Method



Naive Bisection Method

```
% Bisection Method

%Input: f(x) continuous on [a, b]
%       f(a) * f(b) < 0

%Output: p in (a, b) so f(p) = 0.


fa = f(a);
fb = f(b);

repeat
   c = (a+b)/2;
   fc = f(c);
   if (fc ==0)
      p = c;
      return;
   end
   if (fc * fa < 0)
      b = c;
   else
      a = c;
   end
end
```

```
function [x, out] = bisect(FunFcnIn, Intv, params)

TOL  = params.tol;
NO   = params.MaxIt;
[FunFcn,msg] = fcnchk(FunFcnIn,0);
if ~isempty(msg)
    error('InvalidFUN',msg);
end
a     = Intv.a;
b     = Intv.b;
fa    = sign(FunFcn(a));
fb    = sign(FunFcn(b));
if (fa*fb >0)
    error('Initial Interval may not contain root',msg);
end
if a>=b
    error('a must be smaller than b',msg);
end

It = 0;
out.x =[a;b];
out.f =[FunFcn(a);FunFcn(b)];
while (It <= NO)
   c = (a+b)/2;
   out.x = [out.x;c];
   out.f =[out.f;FunFcn(c)];
   fc = sign(FunFcn(c));
   if (fc ==0)
      x = c;
      out.flg = 0;
      out.it  = It;
      return;
   end
   if (fc * fa < 0)
      b = c;
   else
      a = c;
   end
   if (abs(b-a)<=TOL)
      x = (a+b)/2;
      out.flg = 0;
      out.it  = It;
      return;
   end
   It = It + 1;
end
out.flg =1;
out.it = NO;
x = (a+b)/2;
```

Theorem 2.1 Suppose that $f \in C[a, b]$ and $f(a) \cdot f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with

$$|p_n - p| \le \frac{b-a}{2^n}, \quad \text{when} \quad n \ge 1. \qquad \blacksquare$$

Proof of Thm 2.1

Assume that $f(p_n) \neq 0$ for all n.

- By construction

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1.$$

  Thus sequences $\{a_n\}$ and $\{b_n\}$ monotonically converge to limits $a_\infty \leq b_\infty$, respectively.
- Since $f(a_n) \cdot f(b_n) < 0$ for all n, it follows that $f(a_\infty) \cdot f(b_\infty) \leq 0$, and thus a root $p \in [a_\infty, b_\infty] \subset [a_n, b_n]$ exists.
- Since $p_n = \frac{a_n + b_n}{2}$, it follows that $|p_n - p| \leq \frac{b_n - a_n}{2}$.
- By construction

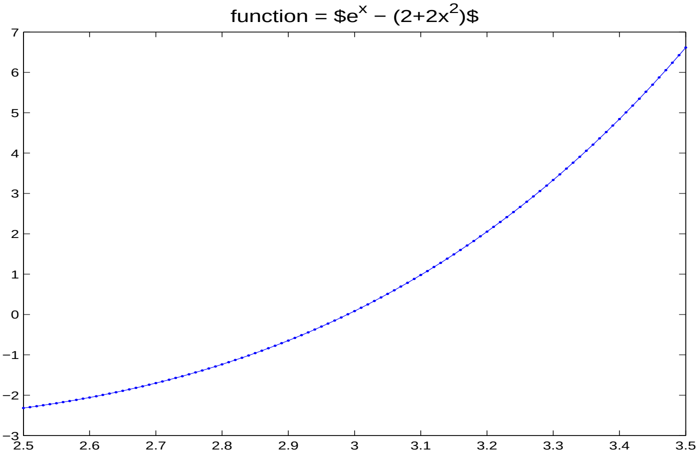
$$b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2} = \frac{b_{n-2} - a_{n-2}}{2^2} = \cdots = \frac{b_1 - a_1}{2^{n-1}} = \frac{b - a}{2^{n-1}}.$$

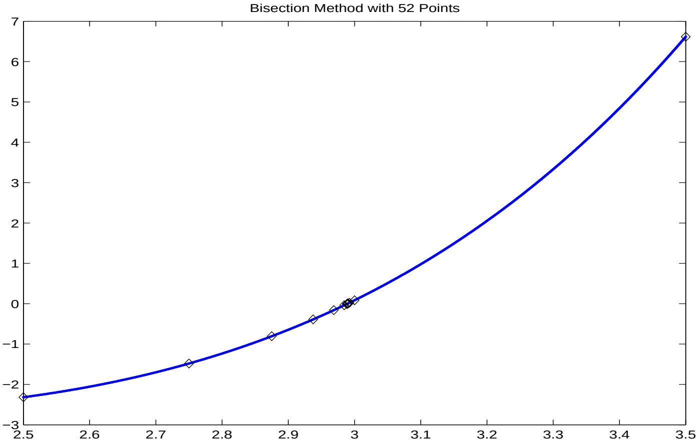
- Put together,

$$|p_n - p| \leq \frac{b - a}{2^n}.$$

- In fact, $a_\infty = b_\infty = p$.

Example Function with Root





Bisection Method with 52 Points
7
6
5
4
3
2
1
0
−1
−2
−3
2.5
2.6
2.7
2.8
2.9
3
3.1
3.2
3.3
3.4
3.5

Fixed Point Iteration

The number p is a **fixed point** for a given function g if $g(p) = p$.

- Given a root-finding problem $f(p) = 0$, we can define functions g with a fixed point at p in a number of ways, for example, as

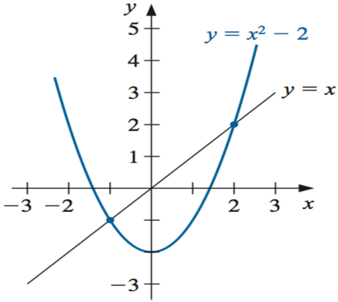
$$g(x) = x - f(x) \quad \text{or as} \quad g(x) = x + 3f(x).$$

- Conversely, if the function g has a fixed point at p, then the function defined by

$$f(x) = x - g(x)$$

has a zero at p.

Fixed Point Example



Fixed Point Theorem (I)

Theorem 2.3

(i) If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has at least one fixed point in $[a,b]$.

(ii) If, in addition, $g'(x)$ exists on (a,b) and a positive constant $k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a,b),$$

then there is exactly one fixed point in $[a,b]$. (See Figure 2.4.) ■

Proof of Thm 2.3

- If $g(a) = a$ or $g(b) = b$,then g has a fixed point at an endpoint.
- Otherwise, $g(a) > a$ and $g(b) < b$. The function $h(x) = g(x) - x$ is continuous on $[a, b]$, with

$$h(a) = g(a) - a > 0 \quad \text{and} \quad h(b) = g(b) - b < 0.$$

- This implies that there exists $p \in (a, b)$, $h(p) = 0$.
- $g(p) - p = 0$, or $p = g(p)$.

If $|g'(x)| \leq k < 1$ for all x in (a, b), and p and q are two distinct fixed points in $[a, b]$. Then a number ξ exists

$$\frac{g(p) - g(q)}{p - q} = g'(\xi) < 1.$$

So

$$1 = \frac{p - q}{p - q} = \frac{g(p) - g(q)}{p - q} = g'(\xi) < 1.$$

This contradiction implies uniqueness of fixed point.

Fixed Point Iteration

Given initial approximation p_0, define *Fixed Point Iteration*

$$p_n = g(p_{n-1}), \quad n = 1, 2, \cdots,$$

If iteration converges to p, then

$$p = \lim_{n \to \infty} p_n = \lim_{n \to \infty} g(p_{n-1}) = g(p).$$

Fixed Point Theorem (II)

Theorem 2.4 **(Fixed-Point Theorem)**

Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, for all x in $[a,b]$. Suppose, in addition, that g' exists on (a,b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a,b).$$

Then for any number p_0 in $[a,b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

converges to the unique fixed point p in $[a,b]$. ■

Fixed Point Example $x - x^2 = 0$: no convergence

$$
\begin{aligned}
g(x) &= x^2 \in [1, \infty] \quad \text{for} \quad x \in [1, \infty], \\
|g'(x)| & \quad \text{unbounded in} [1, \infty].
\end{aligned}
$$

```
>> n = 200;
>> x = zeros(n,1);
>> x(1) = 2;
>> for k=2:n
x(k) = x(k-1)*x(k-1);
end
>> disp([x(n)])
Inf
```

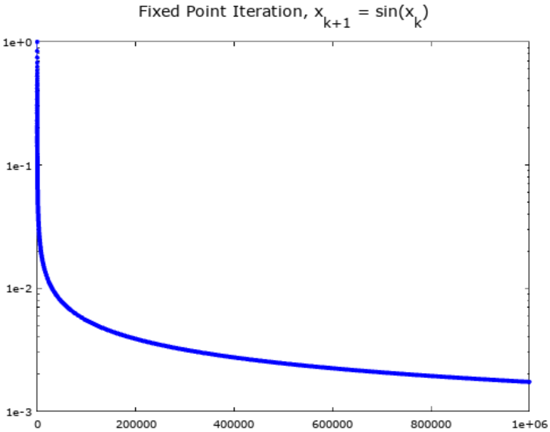
Fixed Point Example $x - \sin(x) = 0$: slow convergence

$$
\begin{aligned}
g(x) &= \sin(x) \in [-1,1] \quad \text{for} \quad x \in [-1,1], \\
|g'(x)| &\le 1 \in [-1,1].
\end{aligned}
$$

```
>> n = 1000000;
>> x = zeros(n,1);
>> x(1) = 1;
>> for k=2:n
x(k) = sin(x(k-1));
end
>> semilogy(abs(x),'b.-')
>> title('Fixed Point Iteration, x_{k+1} = sin(x_k)', 'FontSize', 14)
```

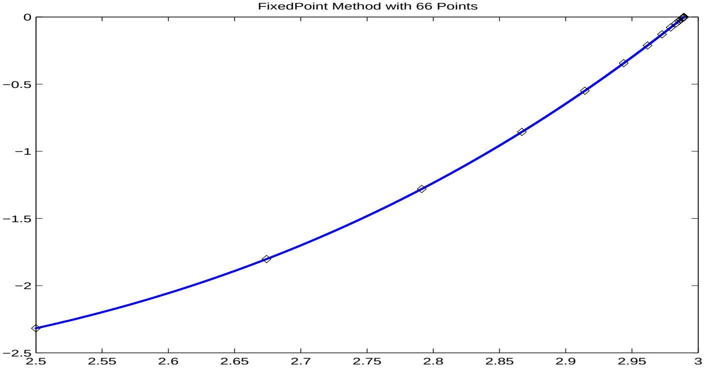
Fixed Point Example $x - \sin(x) = 0$: slow convergence

$$
\begin{aligned}
g(x) &= \sin(x) \in [-1, 1] \quad \text{for} \quad x \in [-1, 1], \\
|g'(x)| &\leq 1 \in [-1, 1].
\end{aligned}
$$



Fixed Point Example $x - \log(2+2x^2) = 0$: normal convergence

$$
\begin{aligned}
g(x) &= \log(2+2x^2) \in [2,3] \quad \text{for} \quad x \in [2,3], \\
|g'(x)| &\le \frac{4}{5} \in [2,3].
\end{aligned}
$$



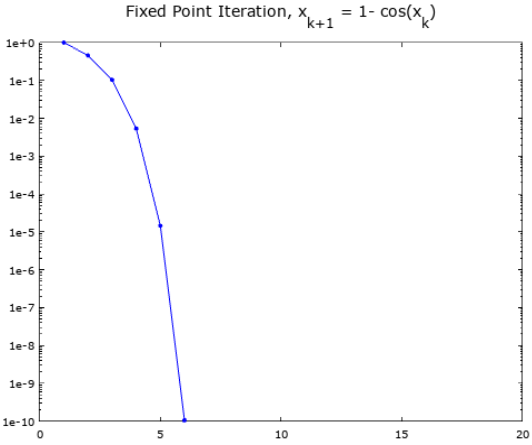
Fixed Point Example $x - (1 - \cos(x)) = 0$: fast convergence

$$\begin{aligned} g(x) &= 1 - \cos(x) \in [-1, 1] \quad \text{for} \quad x \in [-1, 1], \\ |g'(x)| &\approx \quad \text{with.} \end{aligned}$$

```
>>
>> n=20;
>> x = zeros(n,1);
>> x(1) = 1;
>> for k=2:n
x(k) = 1- cos(x(k-1));
end
>> semilogy(abs(x),'b.-')
warning: axis: omitting non-positive data in log plot
```

Fixed Point Example $x - (1 - \cos(x)) = 0$: fast convergence

$$
\begin{aligned}
g(x) &= 1 - \cos(x) \in [-1, 1] \quad \text{for} \quad x \in [-1, 1], \\
|g'(x)| &\leq 1 \in [-1, 1].
\end{aligned}
$$



Fixed Point Theorem (II)

Theorem 2.4 **(Fixed-Point Theorem)**

Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, for all x in $[a,b]$. Suppose, in addition, that g' exists on (a,b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a,b).$$

Then for any number p_0 in $[a,b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

converges to the unique fixed point p in $[a,b]$. ■

Proof of Thm 2.4

- A unique fixed point $p \in [a, b]$ exists.
- $$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(\xi_n)(p_{n-1} - p)| \leq k|p_{n-1} - p|$$
- $$|p_n - p| \leq k|p_{n-1} - p| \leq k^2|p_{n-2} - p| \leq \cdots \leq k^n|p_0 - p|.$$
- Since
$$\lim_{n \to \infty} k^n = 0,$$
$\{p_n\}_{n=0}^{\infty}$ converges to p.

Newton's Method for solving $f(p) = 0$

- Suppose that $f \in C^2[a, b]$.
- Let $p_0 \in [a, b]$ be an approximation to p with

$$f'(p_0) \neq 0, \quad \text{and} \quad |p - p_0| \quad \text{"small"}.$$

- Taylor expand $f(x)$ at $x = p$:

$$0 = f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p)).$$

"Solve" for p:

$$\begin{aligned} p &= p_0 - \frac{f(p_0)}{f'(p_0)} - \frac{(p - p_0)^2}{2f'(p_0)} f''(\xi(p)) \\ &\approx p_0 - \frac{f(p_0)}{f'(p_0)} \stackrel{def}{=} p_1. \end{aligned}$$

Newton's Method: $p_{k+1} = p_k - \frac{f(p_k)}{f'(p_k)}, \quad k = 0, 1, \cdots$

Newton's Method for solving $f(p) = 0$

$$\begin{aligned} p &= p_0 - \frac{f(p_0)}{f'(p_0)} - \frac{(p-p_0)^2}{2f'(p_0)} \\ &\approx p_0 - \frac{f(p_0)}{f'(p_0)}\, f''(\xi(p)) \stackrel{def}{=} p_1. \end{aligned}$$

- If p_0 "close to" p, we can expect fast convergence.
- Best hope in practice: p_0 "not too far from" p. Newton's method may or may not converge.
- If Newton's method converges, it converges quickly.

Geometry of Newton's Method

- Taylor expand $f(x)$ at $x = p$:

$$0 = f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p)).$$

- Replace $f(x)$ by a straight line:

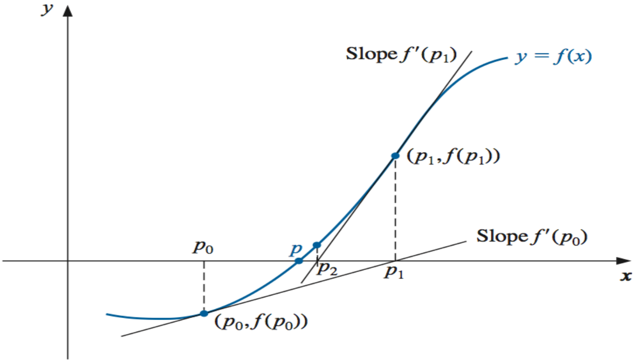
$$f(p_0) + (p - p_0)f'(p_0) \approx 0.$$

- $$p \approx p_0 - \frac{f(p_0)}{f'(p_0)}$$

  is the horizontal intercept of straight line

$$y = f(p_0) + (x - p_0)f'(p_0)$$

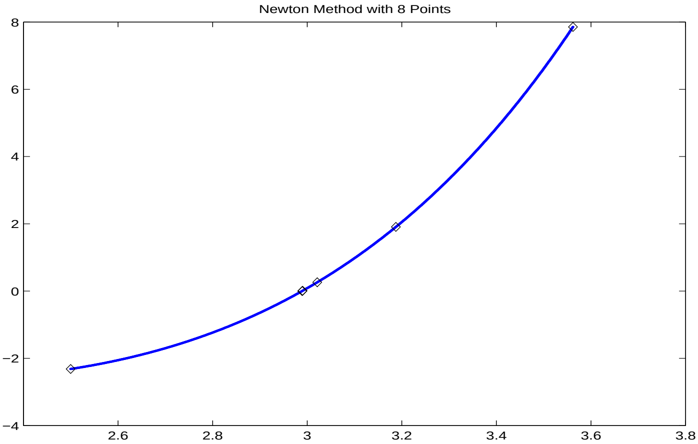
Newton Method



```
function [fun, dfun, x, out] = NewtonMethod(Fun, dFun,x0, params)
%
[FunFcn,msg] = fcnchk(Fun,0);
if ~isempty(msg)
   error('InvalidFUN',msg);
end
[dFunFcn,msg] = fcnchk(dFun,0);
if ~isempty(msg)
   error('InvalidFUN',msg);
end

out.flg = 1;

x(1)  = x0;
N     = params.MaxIt;
tol   = params.tol;
out.x = [];
out.f =[];

for k = 1:N
   fun(k)  = FunFcn(x(k));
   dfun(k) = dFunFcn(x(k));
   out.x   = [out.x;x(k)];
   out.f   = [out.f;fun(k)];
   if (abs(fun(k)) < tol)
       out.flg = 0;
       out.it = k;
       return;
   end
   if (dfun(k) == 0)
       out.it = k;
       return;
   end
   x(k+1) = x(k) - fun(k)/dfun(k);
end
```

Theorem 2.6 Let $f \in C^2[a, b]$. If $p \in (a, b)$ is such that $f(p) = 0$ and $f'(p) \neq 0$, then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial approximation $p_0 \in [p - \delta, p + \delta]$. ■

Newton Method for $f(x) = e^x - (2 + 2x^2)$



Computing square root with Newton's Method

- Given $a > 0$, $p \stackrel{def}{=} \sqrt{a}$ is positive root of equation

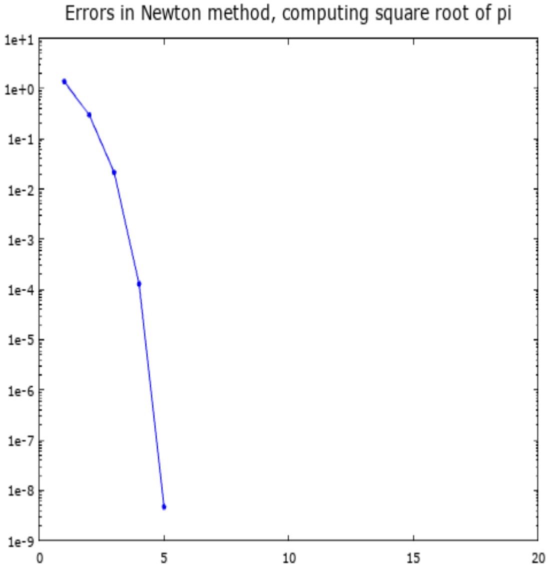
$$f(x) \stackrel{def}{=} x^2 - a = 0.$$

- Newton's Method

$$p_{k+1} = p_k - \frac{p_k^2 - a}{2p_k} = \frac{1}{2}\left(p_k + \frac{a}{p_k}\right), k = 0, 1, 2, \cdots,$$

- Newton's Method is well defined for any $p_0 > 0$.

Newton Method for square root

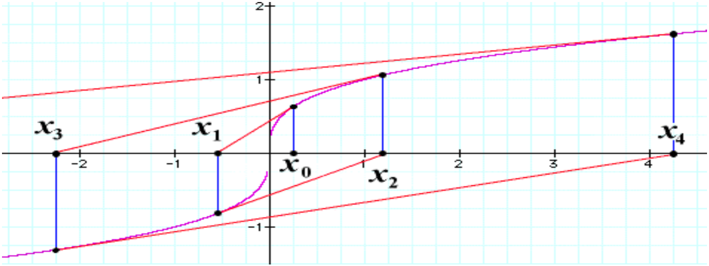


Theorem 2.6 Let $f \in C^2[a, b]$. If $p \in (a, b)$ is such that $f(p) = 0$ and $f'(p) \neq 0$, then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial approximation $p_0 \in [p - \delta, p + \delta]$. ■

Proof of Theorem 2.6

- Newton's method is fixed point iteration $p_n = g(p_{n-1}), g(x) = x - \frac{f(x)}{f'(x)}$.
- Since $f'(p) \neq 0$, there exists an interval $[p-\delta_1, p+\delta_1] \subset [a,b]$ on which $f'(x) \neq 0$. Thus, $g(x)$ is defined on $[p-\delta_1, p+\delta_1]$.
- $$g'(x) = 1 - \frac{f'(x)\,f'(x) - f(x)\,f''(x)}{(f'(x))^2} = \frac{f(x)\,f''(x)}{(f'(x))^2} \in C[p-\delta_1, p+\delta_1].$$
- Since $g'(p) = 0$, there exists $0 < \delta < \delta_1$ so that
$$|g'(x)| \leq \kappa \quad (= \frac{1}{2}), \quad \text{for all} \quad x \in [p-\delta, p+\delta].$$
- If $x \in [p-\delta, p+\delta]$, then
$$|g(x) - p| = |g(x) - g(p)| = |g'(\xi)(x-p)| \leq \kappa|x-p| \leq |x-p|.$$
Therefore $g(x) \in [p-\delta, p+\delta]$.
- $\{p_n\}$ converges to p by Fixed Point Theorem.

Newton Method Divergence Example: $f(x) = x^{1/3}$



Secant Method: Poor man's Newton Method

Motivation

- Newton method style of fast convergence
- Avoid need for derivative calculations.

Approach

- Newton method: $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$.
- Replace $f'(p_n)$ by its cheap approximation

$$f'(p_n) = \lim_{x \to} \frac{f(p_n) - f(x)}{p_n - x} \approx \frac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}}.$$

- Secant method

$$p_{n+1} = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}, n = 1, 2, \cdots .$$

Secant Method: Geometry

- "Approximate" $f(x)$ by a straight line

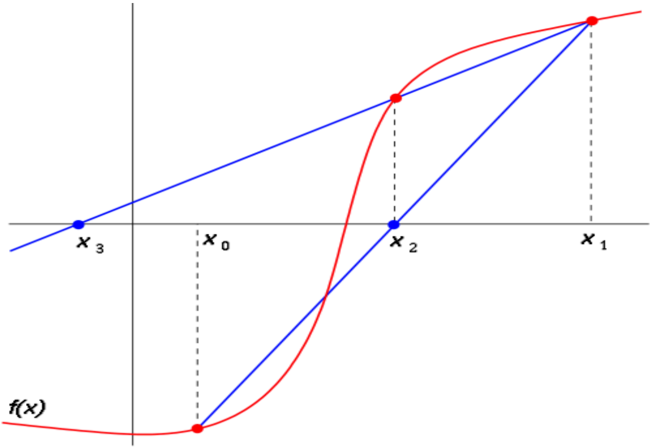
$$f(x) \approx \frac{(x - p_0)f(p_1) - (x - p_1)f(p_0)}{p_1 - p_0}.$$

  Both $f(x)$ and straight line go through points $(p_0, f(p_0)), (p_1, f(p_1))$.

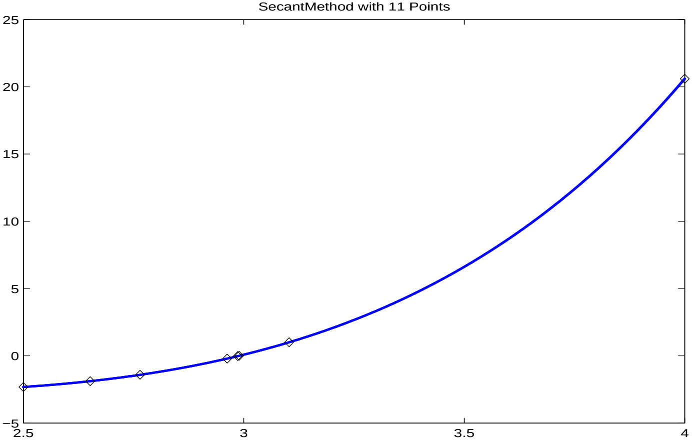
- Let approximate root p_2 be the x-intercept of the straight line

$$p_2 = \frac{p_0 f(p_1) - p_1 f(p_0)}{f(p_1) - f(p_0)} = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)}.$$

Secant Method: Illustration



Fixed point for $g(x) = log(2 + 2x^2)$



Performance: number of iterations vs. error in the solution

- Function to be considered

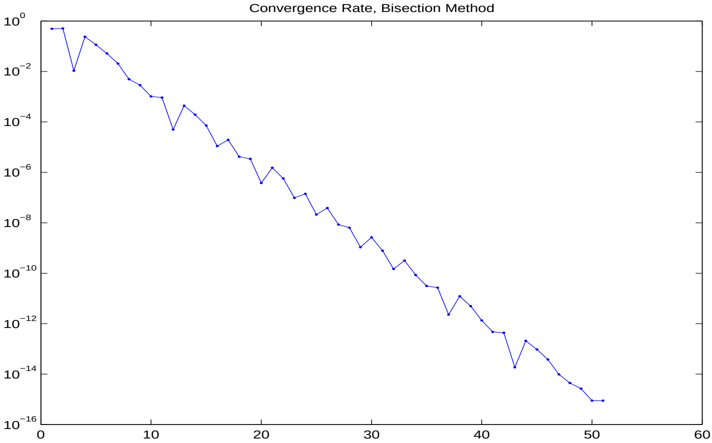
$$g(x) = log(2 + 2x^2), \quad f(x) = x - g(x) = x - log(2 + 2x^2).$$

- Root p of f (i.e., $f(p) = 0$)

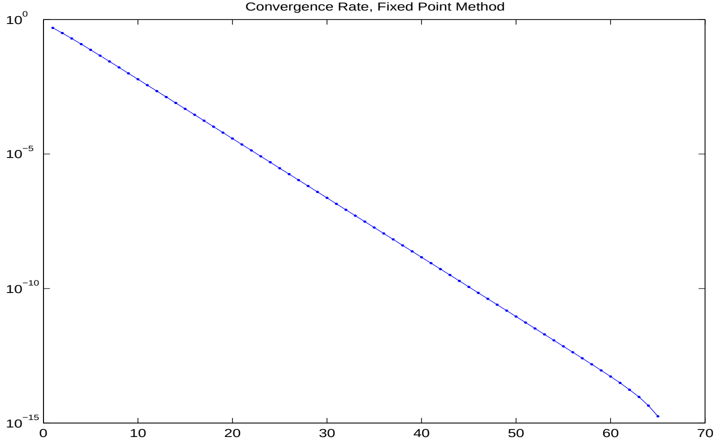
$$p = 2.98930778246493e + 00.$$

- Bisection Method
- Fixed Point Iteration
- Newton's Method
- Secant Method

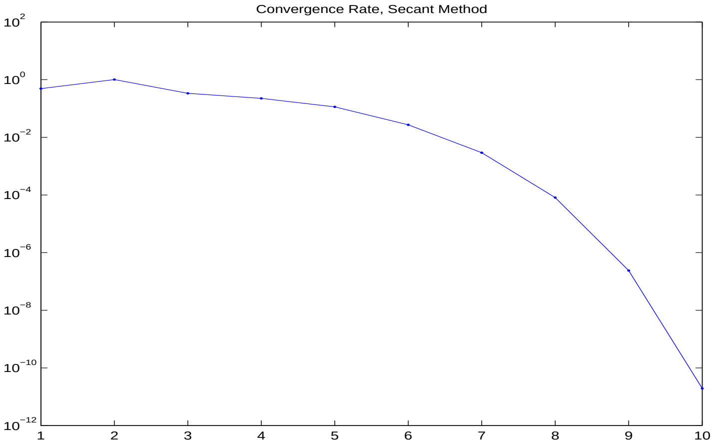
Bisection Method Order of Convergence



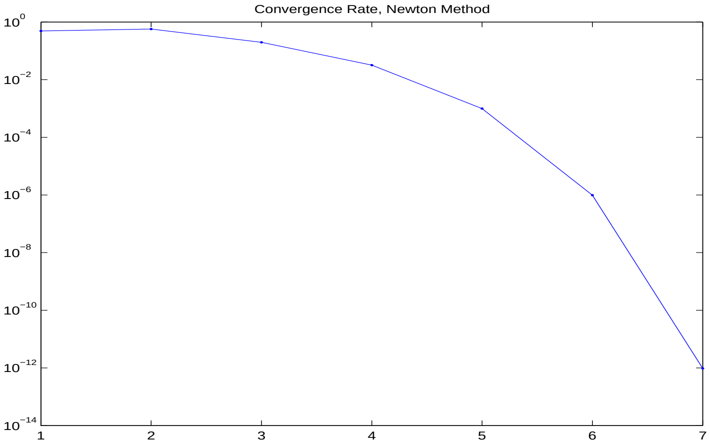
Fixed Point Iteration Order of Convergence



Secant Method Order of Convergence



Newton Method Order of Convergence



Order of convergence

Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p, with $p_n \neq p$ for all n. If positive constants λ and α exist with

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda,$$

then $\{p_n\}_{n=0}^{\infty}$ converges to p of order α, with asymptotic error constant λ.

Linear and Quadratic Order of convergence

(i) If $\alpha = 1$ (and $\lambda < 1$), the sequence is linearly convergent.

(ii) If $\alpha = 2$, the sequence is quadratically convergent.

Recall rate of convergence: the Big O

Suppose $\{\beta_n\}_{n=1}^{\infty}$ is a sequence known to converge to zero, and $\{\alpha_n\}_{n=1}^{\infty}$ converges to a number α. If a positive constant K exists with

$$|\alpha_n - \alpha| \leq K|\beta_n|, \quad \text{for large } n,$$

then we say that $\{\alpha_n\}_{n=1}^{\infty}$ converges to α with **rate, or order, of convergence** $O(\beta_n)$. (This expression is read "big oh of β_n".) It is indicated by writing $\alpha_n = \alpha + O(\beta_n)$. ■

the Big $O()$ = rate of convergence

Recall rate of convergence: the Big O

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Linear and Quadratic Order of convergence

- Suppose that $\{p_n\}_{n=1}^{\infty}$ is linearly convergent to 0,

$$\lim_{n\to\infty} \frac{|p_{n+1}|}{|p_n|} = 0.5, \quad \text{or roughly} \quad \frac{|p_{n+1}|}{|p_n|} \approx 0.5,$$

hence $\boxed{p_n \approx (0.5)^n |p_0|.}$

- Suppose that $\{\tilde{p}_n\}_{n=1}^{\infty}$ is quadratically convergent to 0,

$$\lim_{n\to\infty} \frac{|\tilde{p}_{n+1}|}{|\tilde{p}_n|^2} = 0.5, \quad \text{or roughly} \quad \frac{|\tilde{p}_{n+1}|}{|\tilde{p}_n|^2} \approx 0.5.$$

But now

$$|\tilde{p}_n| \approx 0.5|\tilde{p}_{n-1}|^2 \approx (0.5)[0.5|\tilde{p}_{n-2}|^2]^2 = (0.5)^3|\tilde{p}_{n-2}|^4$$

$$\approx (0.5)^3[(0.5)|\tilde{p}_{n-3}|^2]^4 = (0.5)^7|\tilde{p}_{n-3}|^8$$

$$\approx \cdots \approx (0.5)^{2^n-1}|\tilde{p}_0|^{2^n}.$$

Linear and Quadratic Order of convergence

n	Linear Convergence Sequence $\{p_n\}_{n=0}^{\infty}$ $(0.5)^n$	Quadratic Convergence Sequence $\{\tilde{p}_n\}_{n=0}^{\infty}$ $(0.5)^{2^n-1}$
1	5.0000×10^{-1}	5.0000×10^{-1}
2	2.5000×10^{-1}	1.2500×10^{-1}
3	1.2500×10^{-1}	7.8125×10^{-3}
4	6.2500×10^{-2}	3.0518×10^{-5}
5	3.1250×10^{-2}	4.6566×10^{-10}
6	1.5625×10^{-2}	1.0842×10^{-19}
7	7.8125×10^{-3}	5.8775×10^{-39}

Linear and Quadratic Order of convergence

Linear convergence: one more accurate bit per iteration
Quadratic convergence: double # of correct bits per iteration.