

General linear equations

$$\begin{aligned}E_1 : \quad a_{11}x_1 &+ a_{12}x_2 &\cdots &+ a_{1n}x_n &= b_1, \\E_2 : \quad a_{21}x_1 &+ a_{22}x_2 &\cdots &+ a_{2n}x_n &= b_2, \\&\vdots &&\vdots &&\vdots \\E_n : \quad a_{n1}x_1 &+ a_{n2}x_2 &\cdots &+ a_{nn}x_n &= b_n,\end{aligned}$$

General linear equations

$$E_1 : \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1,$$

$$E_2 : \quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2,$$

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$$E_n : \quad a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n,$$

$$A \stackrel{\text{def}}{=} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} \stackrel{\text{def}}{=} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \quad \mathbf{x} \stackrel{\text{def}}{=} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

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 \vdots \vdots \vdots

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- equation E_j maps to **row** j of A : $(a_{j1}, a_{j2}, \dots, a_{jn})$, $1 \leq j \leq n$.

$$\begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{pmatrix}, \quad 1 \leq k \leq n.$$

- variable x_k relates to **column** k of A :

- will concentrate on A for now.

Gifts from Math God in GE (I), assuming $a_{11} \neq 0$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \text{first column} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}.$$

- elimination in rows E_2 through E_n , on first column:

$$l_{j1} \stackrel{\text{def}}{=} \frac{a_{j1}}{a_{11}}, \quad (E_j - l_{j1} E_1) \rightarrow (E_j), \quad 2 \leq j \leq n.$$

new $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix},$

where $\mathbf{a}_{jk} = a_{jk} - \frac{a_{j1}}{a_{11}} a_{1k}, \quad 2 \leq j \leq n, \quad 2 \leq k \leq n.$

Gifts from Math God in GE (II), assuming $a_{11} \neq 0$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \text{new } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix},$$

where $\mathbf{a}_{jk} = a_{jk} - l_{j1} a_{1k}$, $l_{j1} = \frac{a_{j1}}{a_{11}}$, $2 \leq j \leq n$, $2 \leq k \leq n$.

- ▶ *Gift #1: Matrix-matrix product connection*

$$A = L_1 \cdot (\text{new } A), \quad \text{where } L_1 = \begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & & \ddots & \\ l_{n1} & & & 1 \end{pmatrix}.$$

Gifts from Math God in GE (III), assuming $a_{11} \neq 0$

$$A = L_1 \cdot (\text{new } A) = \begin{pmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ \vdots & & \ddots & & \\ l_{n1} & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix},$$

Gifts from Math God in GE (III), assuming $a_{11} \neq 0$

$$A = L_1 \cdot (\text{new } A) = \begin{pmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ \vdots & & \ddots & & \\ l_{n1} & & & 1 & \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix},$$

- ▶ new step of elimination, assuming $\mathbf{a}_{22} \neq 0$,

$$\begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & & & & \\ l_{32} & 1 & & & \\ \vdots & & \ddots & & \\ l_{n2} & & & 1 & \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a}_{22} & \mathbf{a}_{23} & \cdots & \mathbf{a}_{2n} \\ 0 & \hat{a}_{33} & \cdots & \hat{a}_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \hat{a}_{n3} & \cdots & \hat{a}_{nn} \end{pmatrix}$$

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$$\begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & & & & \\ l_{32} & 1 & & & \\ \vdots & & \ddots & & \\ l_{n2} & & & 1 & \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a}_{22} & \mathbf{a}_{23} & \cdots & \mathbf{a}_{2n} \\ 0 & \hat{a}_{33} & \cdots & \hat{a}_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \hat{a}_{n3} & \cdots & \hat{a}_{nn} \end{pmatrix}$$

- ▶ Gift #2: "free" matrix-matrix product

$$A = \begin{pmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ l_{31} & l_{32} & 1 & & \\ \vdots & \vdots & & \ddots & \\ l_{n1} & l_{n2} & & & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & \mathbf{a}_{22} & \mathbf{a}_{23} & \cdots & \mathbf{a}_{2n} \\ 0 & 0 & \hat{a}_{33} & \cdots & \hat{a}_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \hat{a}_{n3} & \cdots & \hat{a}_{nn} \end{pmatrix}.$$

Gifts from Math God in GE (IV), GE=LU factorization

- GE for $n = 2$:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & \\ l_{21} & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} \\ 0 & \mathbf{a}_{22} \end{pmatrix}$$
$$\stackrel{\text{def}}{=} L \cdot U = \begin{pmatrix} \triangle & \\ & \end{pmatrix} \cdot \begin{pmatrix} & \\ \triangle & \end{pmatrix},$$

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$$\stackrel{\text{def}}{=} L \cdot U = \begin{pmatrix} \triangle & \\ & \end{pmatrix} \cdot \begin{pmatrix} & \\ \diagdown & \triangle \end{pmatrix},$$

- GE for $n \geq 3$:

$$\begin{aligned} A &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \\ \mathbf{I} & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix}, \quad \left(\mathbf{I} \stackrel{\text{def}}{=} \begin{pmatrix} l_{21} \\ \vdots \\ l_{n1} \end{pmatrix} \right). \end{aligned}$$

Gifts from Math God in GE (V), GE=LU factorization

- ▶ Induction hypothesis:

$$\begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} = \mathbf{L} \cdot \mathbf{U} = \begin{pmatrix} \text{triangle} \\ \vdots \\ \text{triangle} \end{pmatrix} \cdot \begin{pmatrix} \text{square} \\ \vdots \\ \text{square} \end{pmatrix}$$

Gifts from Math God in GE (V), GE=LU factorization

- ▶ Induction hypothesis:

$$\begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} = \mathbf{L} \cdot \mathbf{U} = \begin{pmatrix} \text{triangle} \end{pmatrix} \cdot \begin{pmatrix} \text{triangle} \end{pmatrix}$$

- ▶ *Gift #3: LU factorization*

$$A = \begin{pmatrix} 1 & & \\ I & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & (a_{12} & \cdots & a_{1n}) \\ \mathbf{0} & \mathbf{L} \mathbf{U} \end{pmatrix},$$

$$= \begin{pmatrix} 1 & & \\ I & \mathbf{L} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & (a_{12} & \cdots & a_{1n}) \\ \mathbf{0} & \mathbf{U} \end{pmatrix}$$

$$\stackrel{\text{def}}{=} L \cdot U = \begin{pmatrix} \text{triangle} \end{pmatrix} \cdot \begin{pmatrix} \text{triangle} \end{pmatrix}$$

Gaussian Elimination as LU factorization

- In Gaussian Elimination: for $s = 1, \dots, n - 1$,

$$l_{js} = \frac{a_{js}}{a_{ss}}, \quad 1 + s \leq j \leq n,$$

$$a_{jk} \overset{\text{overwrite}}{=} a_{jk} - l_{js} a_{sk}, \quad 1 + s \leq j, k \leq n.$$

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- A becomes upper-triangular after GE:

$$A \xrightarrow{GE} \left(\begin{array}{cccc} & & & \\ & \diagdown & & \\ & & \diagdown & \\ & & & \end{array} \right) \stackrel{\text{def}}{=} U \xrightarrow{\text{notation}} \left(\begin{array}{cccc} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{22} & \cdots & & u_{2n} \\ \ddots & & \vdots & \\ u_{nn} & & & \end{array} \right).$$

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- A becomes upper-triangular after GE:

$$A \xrightarrow{GE} \begin{pmatrix} & & \\ & \ddots & \\ & & 1 \end{pmatrix} \stackrel{\text{def}}{=} U \xrightarrow{\text{notation}} \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{22} & \cdots & & u_{2n} \\ \ddots & & & \vdots \\ & & & u_{nn} \end{pmatrix}.$$

- LU factorization

$$A = L \cdot U, \quad \text{where } L \stackrel{\text{def}}{=} \begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & \vdots & \ddots & \\ l_{n1} & l_{n2} & \cdots & 1 \end{pmatrix} = \begin{pmatrix} & & \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

Gaussian Elimination as LU factorization, example

$$A = \begin{pmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{pmatrix} \in \mathbf{R}^{4 \times 4}$$

$$\begin{aligned} A &= \begin{pmatrix} 1 & & & \\ 2 & 1 & & \\ 3 & & 1 & \\ -1 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 3 \\ -1 & -1 & -1 & -5 \\ -4 & -1 & -7 & \\ 3 & 3 & 2 & \end{pmatrix} \\ &= \begin{pmatrix} 1 & & & \\ 2 & 1 & & \\ 3 & 4 & 1 & \\ -1 & -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 3 \\ -1 & -1 & -1 & -5 \\ 3 & 13 & & \\ -13 & & & \end{pmatrix}. \end{aligned}$$

Row interchange vs. Permutation Matrix

- ▶ **definition:** A permutation matrix $P = (p_{ij})$ is a matrix obtained by rearranging the rows of the identity matrix I_n .

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- ▶ **3 × 3 example**

$$P_{2,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

- ▶ $P_{2,3} \cdot A$ is A with interchanged rows:

$$P_{2,3} \cdot A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \mathbf{a}_{31} & \mathbf{a}_{32} & \mathbf{a}_{33} \\ \mathbf{a}_{21} & \mathbf{a}_{22} & \mathbf{a}_{23} \end{pmatrix}.$$

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- ▶ Let $P_{k,s}$ be permutation interchanging rows k and s of I_n .
For any $A = (a_{ij}) \in \mathbf{R}^{n \times n}$,
 $P_{k,s} \cdot A$ is A with rows k and s interchanged.

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For any $A = (a_{ij}) \in \mathbf{R}^{n \times n}$,
 $P_{k,s} \cdot A$ is A with rows k and s interchanged.
- ▶ Let P be a permutation, then $P P_{k,s}$ is also a permutation.

Gaussian Elimination with partial pivoting: review

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

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- ▶ for $s = 1, 2, \dots, n-1$:
 - ▶ **pivoting**: choose largest entry in absolute value:

$$\mathbf{piv}_s \stackrel{\text{def}}{=} \operatorname{argmax}_{s \leq j \leq n} |a_{js}|, \quad E_s \leftrightarrow E_{\mathbf{piv}_s}$$

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$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

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- ▶ **eliminating** x_s from E_{s+1} through E_n :

$$l_{js} = \frac{a_{js}}{a_{ss}}, \quad s+1 \leq j \leq n,$$

$$a_{jk} = a_{jk} - l_{js} a_{sk}, \quad s+1 \leq j, k \leq n.$$

GEPP as LU factorization, example

$$A = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix} \in \mathbb{R}^{4 \times 4}$$

$$\begin{aligned} P_{1,2} \cdot A &= \begin{pmatrix} 1 & & 1 & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ -1 & & 1 & \\ 1 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 1 & \\ 0 & 1 & 2 & \\ 1 & 1 & 0 & \end{pmatrix}. \end{aligned}$$

$$P_{1,2} \cdot A = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ -1 & & 1 & \\ 1 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 1 & \\ 0 & 1 & 2 & \\ 1 & 1 & 0 & \end{pmatrix}.$$

$$\begin{aligned} P_{1,3} \cdot \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ & 1 & 2 \\ & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ & 1 & 2 \\ & & 3 \end{pmatrix}. \end{aligned}$$

$$P_{1,2} \cdot A = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ -1 & & 1 & \\ 1 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 1 & \\ 0 & 1 & 2 & \\ 1 & 1 & 0 & \end{pmatrix},$$

$$P_{1,3} \cdot \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & \\ 3 & & \end{pmatrix}.$$

► Permutation

$$P = \begin{pmatrix} 1 & \\ & P_{1,3} \end{pmatrix} \cdot P_{1,2} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & \\ 1 & 1 & \\ & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

$$P_{1,2} \cdot A = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ -1 & & 1 & \\ 1 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 1 & \\ 0 & 1 & 2 & \\ 1 & 1 & 0 & \end{pmatrix},$$

$$P_{1,3} \cdot \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & \\ 1 & & 3 \end{pmatrix}.$$

► LU factorization

$$\begin{aligned} P \cdot A &= \left(P_{1,3} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \quad P_{1,3} \right) \cdot \left(\begin{array}{c|ccc} 1 & (1 & -1 & 2) \\ \hline 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{array} \right) \\ &= \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ -1 & & 1 & \\ 0 & & & 1 \end{pmatrix} \left(\begin{array}{c|ccc} 1 & (1 & -1 & 2) \\ \hline P_{1,3} & \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} \end{array} \right). \end{aligned}$$

$$A = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

$$\begin{aligned} P \cdot A &= \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ -1 & & 1 & \\ 0 & & & 1 \end{pmatrix} \cdot \left(\begin{array}{c} 1 \\ \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & -1 & 1 & \end{pmatrix} \\ \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \end{array} \right) \\ &= \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ -1 & 0 & 1 & \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 1 & 0 & \\ 1 & 2 & & \\ 3 & & & \end{pmatrix} \stackrel{\text{def}}{=} L \cdot U. \end{aligned}$$

GEPP as LU factorization

Theorem: Let $A = (a_{ij}) \in \mathbf{R}^{n \times n}$ be non-singular. Then GEPP computes an LU factorization with permutation matrix P such that

$$P \cdot A = L \cdot U = \begin{pmatrix} & & \\ & \diagdown & \\ & & \end{pmatrix} \cdot \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}.$$

$P \cdot A = L \cdot U$, Proof by Induction

- ▶ GEPP for $n = 2$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

- ▶ **pivoting:**

$$\mathbf{piv}_1 \stackrel{\text{def}}{=} \operatorname{argmax}_{1 \leq j \leq 2} |a_{j1}|, \quad P = \begin{cases} I, & \text{if } \mathbf{piv}_1 = 1, \\ P_{1,2}, & \text{if } \mathbf{piv}_1 = 2. \end{cases}$$

- ▶ **elimination:**

$$P \cdot A = L \cdot U.$$

$P \cdot A = L \cdot U$, Proof by Induction

- GEPP for $n \geq 3$, $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$:
- **pivoting:**

$$\mathbf{piv}_1 \stackrel{\text{def}}{=} \operatorname{argmax}_{1 \leq j \leq n} |a_{j1}|, \quad \overline{P} = \begin{cases} I, & \text{if } \mathbf{piv}_1 = 1, \\ P_{1,\mathbf{piv}_1} & \text{if } \mathbf{piv}_1 \geq 2. \end{cases}$$

- **elimination:**

$$\overline{P} \cdot A = \begin{pmatrix} 1 & & \\ & I_{n-1} & \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix}, \quad \left(I \stackrel{\text{def}}{=} \begin{pmatrix} I_{21} \\ \vdots \\ I_{n1} \end{pmatrix} \right).$$

- ▶ Induction hypothesis:

$$\mathbf{P} \cdot \begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} = \mathbf{L} \cdot \mathbf{U}.$$

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- ▶ putting it together,

$$\begin{aligned}
 \begin{pmatrix} 1 & \\ & \mathbf{P} \end{pmatrix} \cdot \overline{\mathbf{P}} \cdot A &= \begin{pmatrix} 1 & \\ & \mathbf{P} \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} \\
 &= \begin{pmatrix} 1 & & \\ & \mathbf{P} \cdot I & \mathbf{P} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \begin{pmatrix} a_{12} & \cdots & a_{1n} \end{pmatrix} \\ \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} \\
 &= \begin{pmatrix} 1 & & \\ & \mathbf{P} \cdot I & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \begin{pmatrix} a_{12} & \cdots & a_{1n} \end{pmatrix} \\ \mathbf{P} \cdot \begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} & \end{pmatrix}
 \end{aligned}$$

$$\text{Let } P \stackrel{\text{def}}{=} \begin{pmatrix} 1 \\ \mathbf{P} \end{pmatrix} \cdot \overline{P}.$$

► LU magic:

$$\begin{aligned} P \cdot A &= \begin{pmatrix} 1 \\ \mathbf{P} \cdot \mathbf{I} & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & (a_{12} & \cdots & a_{1n}) \\ & \mathbf{L} \cdot \mathbf{U} \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ \mathbf{P} \cdot \mathbf{I} & \mathbf{L} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & (a_{12} & \cdots & a_{1n}) \\ & \mathbf{U} \end{pmatrix} \\ &\stackrel{\text{def}}{=} L \cdot U = \begin{pmatrix} \triangle \\ \vdots \end{pmatrix} \cdot \begin{pmatrix} \square \\ \vdots \end{pmatrix}. \end{aligned}$$

Solving general linear equations with GEPP

$$A\mathbf{x} = \mathbf{b}, \quad P \cdot A = L \cdot U$$

- ▶ interchanging components in \mathbf{b}

$$P \cdot (A\mathbf{x}) = (P \cdot \mathbf{b}), \quad (L \cdot U) \mathbf{x} = (P \cdot \mathbf{b}).$$

- ▶ solving for \mathbf{b} with forward and backward substitution

$$\begin{aligned}\mathbf{x} &= (L \cdot U)^{-1} (P \cdot \mathbf{b}) \\ &= (U^{-1} (L^{-1} (P \cdot \mathbf{b}))).\end{aligned}$$

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Cost Analysis

- ▶ computing $P \cdot A = L \cdot U$: about $2/3n^3$ operations.
- ▶ forward and backward substitution: about $2n^2$ operations.
- ▶ most important to compute $P \cdot A = L \cdot U$ **efficiently**

Strictly Diagonally Dominant (**SDD**) Matrices

- ▶ **Definition:** Matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is **SDD** if

$$|a_{ii}| > \sum_{j=1, j \neq i}^n |a_{ij}| \quad \text{holds for each } i = 1, 2, \dots, n.$$

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- Example I: matrix $A = \begin{pmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{pmatrix}$ is **SDD**.

$$|7| > |2| + |0|, \quad |5| > |3| + |-1|, \quad |-6| > |0| + |5|.$$

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- Example II: matrix $B = \begin{pmatrix} 7 & 5 & 0 \\ 3 & 5 & -1 \\ 0 & -3 & 3 \end{pmatrix}$ is NOT **SDD**.

$$|3| \leq |0| + |-3|.$$

GE on **SDD**: succeeds without pivoting (I)

- ▶ Let $A = (a_{ij}) \in \mathbf{R}^{n \times n}$ be **SDD**, so

$$|a_{11}| > \sum_{j=1, j \neq 1}^n |a_{1j}| \geq 0.$$

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$$A = L_1 \cdot (\text{new } A)$$

$$= \begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & \ddots & \ddots & \\ l_{n1} & & & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix},$$

$$l_{j1} = \frac{a_{j1}}{a_{11}}, \quad \mathbf{a}_{jk} = a_{jk} - \frac{a_{j1}}{a_{11}} a_{1k}, \quad 2 \leq j \leq n, \quad 2 \leq k \leq n.$$

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- only do:** show $\mathbf{A} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix}$ remains **SDD**.

GE on **SDD**: only need to show **A** remains **SDD**

$$|a_{11}| > \sum_{j=1, j \neq 1}^n |a_{1j}|, \quad |a_{ii}| > \sum_{j=1, j \neq i}^n |a_{ij}|, \quad i = 2, \dots, n,$$

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix}, \quad \mathbf{a}_{ij} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j}.$$

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- ▶ for $i = 2, \dots, n$,

$$\sum_{j=2, j \neq i}^n |\mathbf{a}_{ij}| = \sum_{j=2, j \neq i}^n \left| a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j} \right| \leq \left(\sum_{j=2, j \neq i}^n |a_{ij}| \right) + \left| \frac{a_{i1}}{a_{11}} \right| \left(\sum_{j=2, j \neq i}^n |a_{1j}| \right)$$

GE on **SDD**: only need to show **A** remains **SDD**

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- ▶ for $i = 2, \dots, n$,

$$\begin{aligned} \sum_{j=2, j \neq i}^n |\mathbf{a}_{ij}| &= \sum_{j=2, j \neq i}^n \left| a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j} \right| \leq \left(\sum_{j=2, j \neq i}^n |a_{ij}| \right) + \left| \frac{a_{i1}}{a_{11}} \right| \left(\sum_{j=2, j \neq i}^n |a_{1j}| \right) \\ &\stackrel{\text{SDD}}{<} (|a_{ii}| - |a_{i1}|) + \left| \frac{a_{i1}}{a_{11}} \right| (|a_{11}| - |a_{1i}|) = |a_{ii}| - \left| \frac{a_{i1}}{a_{11}} \right| |a_{1i}| \end{aligned}$$

GE on SDD: only need to show \mathbf{A} remains SDD

$$|a_{11}| > \sum_{j=1, j \neq 1}^n |a_{1j}|, \quad |a_{ii}| > \sum_{j=1, j \neq i}^n |a_{ij}|, \quad i = 2, \dots, n,$$

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix}, \quad \mathbf{a}_{ij} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j}.$$

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GE on **SDD**: example

$$A = \begin{pmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{pmatrix} \text{ is } \mathbf{SDD}.$$

$$A = \begin{pmatrix} 1 & & \\ \frac{3}{7} & 1 & \\ 0 & & 1 \end{pmatrix} \cdot \begin{pmatrix} 7 & 2 & 0 \\ \frac{29}{7} & -1 & \\ 5 & -6 & \end{pmatrix} \quad \left[\begin{pmatrix} \frac{29}{7} & -1 \\ 5 & -6 \end{pmatrix} \text{ is } \mathbf{SDD} \right]$$

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