Matrices and vectors

- Matrix and vector

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \overset{def}{=} \left(\, a_{ij} \, \right) \in \mathbf{R}^{m \times n}, \;\; b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathbf{R}^m.$$

Matrix and vectors in linear equations: example

$$
\begin{array}{lrcrcrcrcr}
E_1: & x_1 & + & x_2 & & & + & 3x_4 & = & 4, \\
E_2: & 2x_1 & + & x_2 & - & x_3 & + & x_4 & = & 1, \\
E_3: & 3x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & -3, \\
E_4: & -x_1 & + & 2x_2 & + & 3x_3 & - & x_4 & = & 4.
\end{array}
$$

coefficient matrix $A \stackrel{def}{=} \begin{pmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{pmatrix}$,

unknown vector $\mathbf{x} \stackrel{def}{=} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$,

right hand side vector (RHS) $\mathbf{b} \stackrel{def}{=} \begin{pmatrix} 4 \\ 1 \\ -3 \\ 4 \end{pmatrix}$.

Gaussian Elimination for $A \in \mathbf{R}^{2\times 2}$, $a_{11} \neq 0$.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

$$\begin{aligned}
\ell &= \frac{a_{21}}{a_{11}} \quad \text{(elimination: } (E_2 - \ell E_1) \to (E_2)) \\
a_{22} &= a_{22} - \ell \times a_{12} \quad \text{(matrix overwrite)} \\
b_2 &= b_2 - \ell \times b_1 \quad \text{(RHS overwrite)} \\
x_2 &= \frac{b_2}{a_{22}} \quad \text{(backward substitution)} \\
x_1 &= \frac{b_1 - a_{12} \times x_2}{a_{11}}
\end{aligned}$$

Gaussian Elimination: 2×2 example

- random 2×2 system

$$A = \texttt{randn}\,(2,2)\,, \quad b = \texttt{randn}\,(2,1)\,.$$

- $(1,1)$ becomes inceasingly small in magnitude

$$\begin{aligned} B &= A, \quad B(1,1) = \texttt{randn}/10^k, \quad k = 1, 2, \cdots, 17, \\ x &= B \setminus b, \quad r \stackrel{def}{=} b - B \times x. \end{aligned}$$

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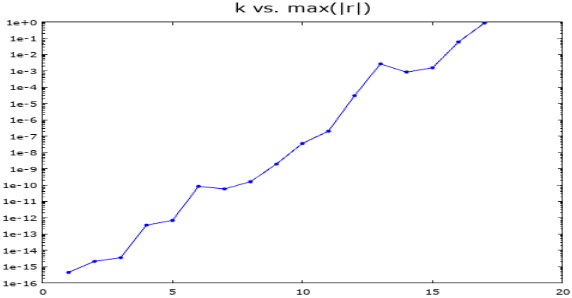
- 

General linear equations

$$
\begin{array}{lcccccccc}
E_1: & a_{11}x_1 & + & a_{12}x_2 & \cdots & + & a_{1n}x_n & = & b_1, \\
E_2: & a_{21}x_1 & + & a_{22}x_2 & \cdots & + & a_{2n}x_n & = & b_2, \\
\vdots & & \vdots & & & & & \vdots & \\
E_n: & a_{n1}x_1 & + & a_{n2}x_2 & \cdots & + & a_{nn}x_n & = & b_n,
\end{array}
$$

General linear equations

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E_1: & a_{11}x_1 & + & a_{12}x_2 & \cdots & + & a_{1n}x_n & = & b_1, \\
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\vdots & & \vdots & & & & & \vdots & \\
E_n: & a_{n1}x_1 & + & a_{n2}x_2 & \cdots & + & a_{nn}x_n & = & b_n,
\end{array}$$

$$A \stackrel{def}{=} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} \stackrel{def}{=} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \quad \mathbf{x} \stackrel{def}{=} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

General linear equations

$$
\begin{array}{lccccccccc}
E_1: & a_{11}x_1 & + & a_{12}x_2 & \cdots & + & a_{1n}x_n & = & b_1, \\
E_2: & a_{21}x_1 & + & a_{22}x_2 & \cdots & + & a_{2n}x_n & = & b_2, \\
\vdots & & \vdots & & & & & \vdots & \\
E_n: & a_{n1}x_1 & + & a_{n2}x_2 & \cdots & + & a_{nn}x_n & = & b_n,
\end{array}
$$

$$
A \stackrel{def}{=} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} \stackrel{def}{=} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \quad \mathbf{x} \stackrel{def}{=} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.
$$

- equation E_j maps to **row** j of A: $\left(a_{j1}, a_{j2}, \cdots, a_{jn}\right)$, $1 \le j \le n$.
- variable x_k relates to **column** k of A: $\begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{pmatrix}$, $1 \le k \le n$.

Gaussian Elimination with partial pivoting (I)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

- equation E_j maps to **row** j of A, x_1 relates to $\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}$.
- **pivoting**: choose largest entry in absolute value:

$$\mathbf{piv} \overset{def}{=} \mathbf{argmax}_{1 \le j \le n} |a_{j1}|, \quad (|a_{\mathbf{piv},1}| = \mathbf{max}_{1 \le j \le n} |a_{j1}|)$$

exchange equations E_1 and $E_{\mathbf{piv}}$ $\quad (E_1 \leftrightarrow E_{\mathbf{piv}}.)$

Gaussian Elimination with partial pivoting (II)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

- elimination of x_1 from E_2 through E_n:

$$\left(E_j - \frac{a_{j1}}{a_{11}} E_1\right) \rightarrow (E_j), \quad \left|\frac{a_{j1}}{a_{11}}\right| \leq 1, \quad 2 \leq j \leq n.$$

- new row j of A

$$\begin{aligned} &\begin{pmatrix} a_{j1} & a_{j2} & \cdots & a_{jn} \end{pmatrix} - \frac{a_{j1}}{a_{11}} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix} \\ &= \begin{pmatrix} 0 & a_{j2} - \frac{a_{j1}}{a_{11}} a_{12} & \cdots & a_{jn} - \frac{a_{j1}}{a_{11}} a_{1n} \end{pmatrix}, \quad 2 \leq j \leq n. \end{aligned}$$

- new j^{th} component of $\mathbf{b}$

$$b_j - \frac{a_{j1}}{a_{11}} b_1.$$

Gaussian Elimination with partial pivoting (III)

$$\textbf{new} \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} \overset{overwrite}{=\!=\!=} A,$$

$$\text{where} \quad \mathbf{a}_{jk} = a_{jk} - \frac{a_{j1}}{a_{11}} a_{1k}, \quad 2 \le j \le n, \quad 2 \le k \le n.$$

$$\textbf{new} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_n \end{pmatrix} \overset{overwrite}{=\!=\!=} \mathbf{b}.$$

$$\text{where} \quad \mathbf{b}_j = b_j - \frac{a_{j1}}{a_{11}} b_1, \quad 2 \le j \le n, \quad 2 \le k \le n.$$

(bold faced entries are computed from elimination process.)

Gaussian Elimination with partial pivoting (IV)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Gaussian Elimination with partial pivoting (IV)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

- repeat same process on E_s for $s = 2, \cdots, n-1$:
  - **pivoting**: choose largest entry in absolute value:

    $$\mathbf{piv} \overset{def}{=} \mathbf{argmax}_{s \le j \le n} |a_{js}|, \quad (|a_{\mathbf{piv},s}| = \mathbf{max}_{1 \le j \le n} |a_{js}|)$$

    exchange equations E_s and $E_{\mathbf{piv}}$ $\quad (E_s \leftrightarrow E_{\mathbf{piv}}.)$

Gaussian Elimination with partial pivoting (IV)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

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    exchange equations E_s and $E_{\mathbf{piv}}$ $\quad (E_s \leftrightarrow E_{\mathbf{piv}}.)$
  - **eliminating** x_s from E_{s+1} through E_n:

    $$\left(E_j - \frac{a_{js}}{a_{ss}} E_s\right) \to (E_j), \quad s+1 \le j \le n.$$

$$\mathbf{a}_{jk} = a_{jk} - \frac{a_{js}}{a_{ss}} a_{sk} \stackrel{overwrite}{=\!=\!=} a_{jk}, \quad s+1 \le j \le n, \quad s+1 \le k \le n,$$

$$\mathbf{b}_j = b_j - \frac{a_{js}}{a_{ss}} b_s \stackrel{overwrite}{=\!=\!=} b_j, \quad s+1 \le j \le n.$$

Gaussian Elimination with partial pivoting: summary

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

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- for $s = 1, 2, \cdots, n-1$:
  - **pivoting**: choose largest entry in absolute value:

$$\mathbf{piv} \stackrel{def}{=} \mathbf{argmax}_{s \leq j \leq n} |a_{js}|, \quad E_s \leftrightarrow E_{\mathbf{piv}}.$$

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  - **eliminating** x_s from E_{s+1} through E_n:

$$\begin{aligned} \mathbf{a}_{jk} &= a_{jk} - \frac{a_{js}}{a_{ss}} a_{sk} \overset{overwrite}{=\!=\!=} a_{jk}, \quad s+1 \le j \le n, \quad s+1 \le k \le n, \\ \mathbf{b}_j &= b_j - \frac{a_{js}}{a_{ss}} b_s \overset{overwrite}{=\!=\!=} b_j, \quad s+1 \le j \le n. \end{aligned}$$

Equations after Gaussian Elimination, backward substitution

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Equations after Gaussian Elimination, backward substitution

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

- for $s = n, n-1, \cdots, 1$:

$$x_s = \frac{b_s - \sum_{k=s+1}^{n} a_{sk} x_k}{a_{ss}}.$$

`matlab command` $x = A\backslash b$.

Gaussian Elimination: cost analysis (I)

- for $s = 1, 2, \cdots, n-1$:
  - **pivoting**: ($n - s + 1$ comparisons)

$$\mathbf{piv} = \mathbf{argmax}_{s \le j \le n} |a_{js}|, \quad E_s \leftrightarrow E_{\mathbf{piv}}.$$

Gaussian Elimination: cost analysis (I)

- for $s = 1, 2, \cdots, n-1$:
  - **pivoting**: $(n - s + 1$ comparisons$)$

$$\mathbf{piv} = \mathbf{argmax}_{s \le j \le n} |a_{js}|, \quad E_s \leftrightarrow E_{\mathbf{piv}}.$$

  - **eliminating** x_s from E_{s+1} through E_n:

$$\begin{aligned} \mathbf{a}_{jk} &= a_{jk} - \frac{a_{js}}{a_{ss}} a_{sk}, \quad s+1 \le j \le n, \quad s+1 \le k \le n, \\ \mathbf{b}_j &= b_j - \frac{a_{js}}{a_{ss}} b_s \quad , \quad s+1 \le j \le n. \end{aligned}$$

Counting operations.

- compute **piv** for each s, and perform swaps $E_s \leftrightarrow E_{\mathbf{piv}}$
- for each s, compute ratios $\frac{a_{js}}{a_{ss}}$ for each $j \ge s+1$.
- for each s, compute $\mathbf{a}_{jk}$ for each pair of $j, k \ge s+1$.
- for each s, compute $\mathbf{b}_j$ for each $j \ge s+1$.

Do not count integer operations; do not count memory costs

Gaussian Elimination: cost analysis (II)

- for $s = 1, 2, \cdots, n-1$:
  - **pivoting**: $(n - s + 1$ comparisons$)$

$$\mathbf{piv} = \mathbf{argmax}_{s \le j \le n} |a_{js}|, \quad E_s \leftrightarrow E_{\mathbf{piv}}.$$

$$(\textit{total comparisons: } \sum_{s=1}^{n-1}(n - s + 1) = \tfrac{n(n+1)}{2} - 1\ )$$

Gaussian Elimination: cost analysis (II)

- for $s = 1, 2, \cdots, n-1$:
  - **pivoting**: $(n - s + 1$ comparisons$)$

$$\mathbf{piv} = \mathbf{argmax}_{s \le j \le n} |a_{js}|, \quad E_s \leftrightarrow E_{\mathbf{piv}}.$$

    (*total comparisons:* $\sum_{s=1}^{n-1}(n-s+1) = \frac{n(n+1)}{2} - 1$)
  - **eliminating** x_s from E_{s+1} through E_n:

$$\begin{aligned} \mathbf{a}_{jk} &= a_{jk} - \frac{a_{js}}{a_{ss}}\, a_{sk}, \quad s+1 \le j \le n, \quad s+1 \le k \le n, \\ \mathbf{b}_j &= b_j - \frac{a_{js}}{a_{ss}}\, b_s \quad, \quad s+1 \le j \le n. \end{aligned}$$

(*total cost for computing* $\left\{\frac{a_{js}}{a_{ss}}\right\}$: $\sum_{s=1}^{n-1}(n-s) = \frac{n(n-1)}{2}$)
(*total cost for computing* $\{\mathbf{a}_{jk}\}$: $\sum_{s=1}^{n-1} 2(n-s)^2 = \frac{n(n-1)(2n-1)}{3}$)
(*total cost for computing* $\{\mathbf{b}_j\}$: $\sum_{s=1}^{n-1} 2(n-s) = n(n-1)$)

Gaussian Elimination: cost analysis (II)

- for $s = 1, 2, \cdots, n-1$:
  - **pivoting**: $(n - s + 1$ comparisons$)$

$$\textbf{piv} = \textbf{argmax}_{s \le j \le n} |a_{js}|, \quad E_s \leftrightarrow E_{\textbf{piv}}.$$

Gaussian Elimination: cost analysis (II)

- for $s = 1, 2, \cdots, n-1$:
  - **pivoting**: ($n - s + 1$ comparisons)

$$\mathbf{piv} = \mathbf{argmax}_{s \le j \le n} |a_{js}|, \quad E_s \leftrightarrow E_{\mathbf{piv}}.$$

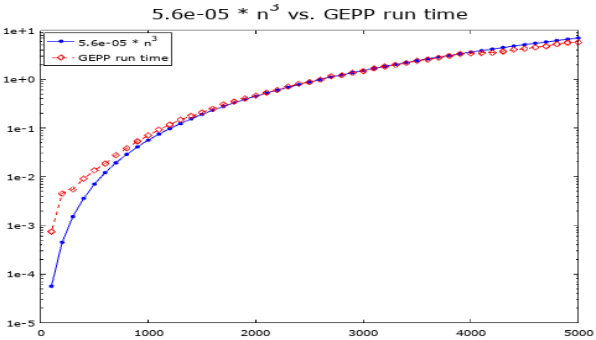
  - **eliminating** x_s from E_{s+1} through E_n:

$$\begin{aligned} \mathbf{a}_{jk} &= a_{jk} - \frac{a_{js}}{a_{ss}}\, a_{sk}, \quad s+1 \le j \le n, \quad s+1 \le k \le n, \\ \mathbf{b}_j &= b_j - \frac{a_{js}}{a_{ss}}\, b_s \quad , \quad s+1 \le j \le n. \end{aligned}$$

  about $2(n-s)^2$ additions and multiplications for each s.

grand total, up to n^3 term: $\frac{2}{3}\, n^3$ additions and multiplications.

Gaussian Elimination: performance on `mac` desktop, in `octave`, $\boxed{\textbf{GEPPruntime} \approx 5.6 \times 10^{-5} \times n^3}$



NO scaled partial pivoting.

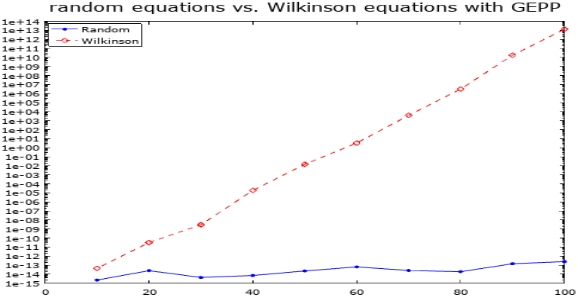
Gaussian Elimination with partial pivoting

- **blue**: $A \in \mathbf{R}^{n \times n}$, $b \in \mathbf{R}^n$ random

Gaussian Elimination with partial pivoting

- **blue**: $A \in \mathbf{R}^{n\times n}$, $b \in \mathbf{R}^n$ random
- **red**: $A \in \mathbf{R}^{n\times n}$ is Wilkinson matrix

$$A = \begin{pmatrix} 1 & & & & 1 \\ -1 & 1 & & & 1 \\ -1 & -1 & 1 & & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 1 \end{pmatrix}, \quad b \in \mathbf{R}^n \quad \text{random.}$$



Gaussian Elimination with complete pivoting, $A \in \mathbf{R}^{2\times 2}$.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

- **pivoting**: choose largest entry in absolute value:

$$(\mathbf{ip}, \mathbf{jp}) \stackrel{def}{=} \mathbf{argmax}_{1\leq i,j\leq 2}|a_{ij}|, \quad (|a_{\mathbf{ip},\mathbf{jp}}| = \mathbf{max}_{1\leq j\leq n}|a_{ij}|)$$

- **exchange** equations E_1 and $E_{\mathbf{ip}}$ $\quad (E_1 \leftrightarrow E_{\mathbf{ip}}.)$
- **exchange** columns/variables 1 and **jp**.
- **perform** Gaussian Elimination without pivoting on the resulting 2×2 system.

Gaussian Elimination with complete pivoting

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Gaussian Elimination with complete pivoting

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

- for $s = 1, 2, \cdots, n-1$:
  - **pivoting**: choose largest entry in absolute value:

$$(\mathbf{ip}, \mathbf{jp}) \stackrel{def}{=} \mathbf{argmax}_{s \leq i,j \leq n} |a_{ij}|, \quad E_s \leftrightarrow E_{\mathbf{ip}}.$$

swap matrix columns/ variables s and **jp**

Gaussian Elimination with complete pivoting

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

- for $s = 1, 2, \cdots, n-1$:
  - **pivoting**: choose largest entry in absolute value:

$$(\mathbf{ip}, \mathbf{jp}) \stackrel{def}{=} \mathbf{argmax}_{s \le i,j \le n} |a_{ij}|, \quad E_s \leftrightarrow E_{\mathbf{ip}}.$$

swap matrix columns/ variables s and **jp**

  - **eliminating** x_s from E_{s+1} through E_n:

$$\mathbf{a}_{jk} = a_{jk} - \frac{a_{js}}{a_{ss}} a_{sk} \stackrel{overwrite}{=\!=\!=} a_{jk}, \quad s+1 \le j \le n, \quad s+1 \le k \le n,$$

$$\mathbf{b}_j = b_j - \frac{a_{js}}{a_{ss}} b_s \stackrel{overwrite}{=\!=\!=} b_j, \quad s+1 \le j \le n.$$

more numerically stable, but too expensive in practice

Matrix Algebra

- **Definition**: Let

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{pmatrix} \stackrel{def}{=} (a_{ij}) \in \mathbf{R}^{n \times m},$$

$$B = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,m} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,m} \end{pmatrix} \stackrel{def}{=} (b_{ij}) \in \mathbf{R}^{n \times m}, \quad \text{then}$$

$$C \stackrel{def}{=} \begin{pmatrix} \alpha\, a_{1,1} + \beta\, b_{1,1} & \alpha\, a_{1,2} + \beta\, b_{1,2} & \cdots & \alpha\, a_{1,m} + \beta\, b_{1,m} \\ \alpha\, a_{2,1} + \beta\, b_{2,1} & \alpha\, a_{2,2} + \beta\, b_{2,2} & \cdots & \alpha\, a_{2,m} + \beta\, b_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha\, a_{n,1} + \beta\, b_{n,1} & \alpha\, a_{n,2} + \beta\, b_{n,2} & \cdots & \alpha\, a_{n,m} + \beta\, b_{n,m} \end{pmatrix}$$

$$= \alpha\, A + \beta\, B \in \mathbf{R}^{n \times m}$$

Matrix Algebra: example

- Let

$$A = \begin{pmatrix} 2 & -1 & 7 \\ 3 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 2 & -8 \\ 0 & 1 & 6 \end{pmatrix} \in \mathbf{R}^{2\times 3}, \quad \text{then}$$

$$C \stackrel{def}{=} A - 2B = \begin{pmatrix} -6 & -5 & 23 \\ 3 & -1 & -12 \end{pmatrix} \in \mathbf{R}^{2\times 3}.$$

Matrix Algebra

Let $A,\ B,\ C \in \mathbf{R}^{n\times m}$, and let $\lambda, \mu \in \mathbf{R}$

(i) $A+B=B+A$,

(ii) $(A+B)+C=A+(B+C)$,

(iii) $A+O=O+A=A$,

(iv) $A+(-A)=-A+A=0$,

(v) $\lambda(A+B)=\lambda A+\lambda B$,

(vi) $(\lambda+\mu)A=\lambda A+\mu A$,

(vii) $\lambda(\mu A)=(\lambda\mu)A$,

(viii) $1A=A$.

Matrix-vector product vs. linear equations

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & \cdots & + & a_{1n}x_n & = & b_1, \\
a_{21}x_1 & + & a_{22}x_2 & \cdots & + & a_{2n}x_n & = & b_2, \\
\vdots & & \vdots & & & \vdots & & \\
a_{n1}x_1 & + & a_{n2}x_2 & \cdots & + & a_{nn}x_n & = & b_n,
\end{array}
$$

Matrix-vector product vs. linear equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & \cdots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & \cdots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & & & \vdots & \\ a_{n1}x_1 & + & a_{n2}x_2 & \cdots & + & a_{nn}x_n & = & b_n, \end{array}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Matrix-vector product vs. linear equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & \cdots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & \cdots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & & \vdots & & \\ a_{n1}x_1 & + & a_{n2}x_2 & \cdots & + & a_{nn}x_n & = & b_n, \end{array}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

$$A\,\mathbf{x} \stackrel{def}{=} \begin{pmatrix} a_{11}x_1 & + & a_{12}x_2 & \cdots & + & a_{1n}x_n \\ a_{21}x_1 & + & a_{22}x_2 & \cdots & + & a_{2n}x_n \\ \vdots & & \vdots & & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & \cdots & + & a_{nn}x_n \end{pmatrix}, \text{ for any } A \in \mathbf{R}^{n\times m}, \mathbf{x} \in \mathbf{R}^m.$$

$$\boxed{\text{equations become: } A\,x = \mathbf{b}}$$

Matrix-vector product is dot product

$$A\mathbf{x} \stackrel{def}{=} \begin{pmatrix} a_{11}x_1 & + & a_{12}x_2 & \cdots & + & a_{1n}x_n \\ a_{21}x_1 & + & a_{22}x_2 & \cdots & + & a_{2n}x_n \\ \vdots & & \vdots & & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & \cdots & + & a_{nn}x_n \end{pmatrix}.$$

$$(A\mathbf{x})_j = \begin{pmatrix} a_{j1}x_1 & + & a_{j2}x_2 & \cdots & + & a_{jn}x_n \end{pmatrix} = \begin{pmatrix} a_{j1} & a_{j2} & \cdots \\ a_{jn} & & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Matrix-vector product, example

$$A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \\ 6 & 4 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

$$A\mathbf{x} = \begin{pmatrix} 3 & 2 \\ -1 & 1 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \\ 14 \end{pmatrix}.$$

Matrix-matrix product

- Let

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \in \mathbf{R}^{n\times m}, \quad B=\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{pmatrix} \in \mathbf{R}^{m\times p}.$$

Matrix-matrix product

- Let

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \in \mathbf{R}^{n\times m}, \ B=\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{pmatrix} \in \mathbf{R}^{m\times p}.$$

- Column partition:

$$B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{pmatrix}, \quad \mathbf{b}_j \stackrel{def}{=} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{pmatrix} \in \mathbf{R}^m, \quad j = 1, \cdots, p.$$

Matrix-matrix product

- Let

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \in \mathbf{R}^{n\times m}, \quad B=\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{pmatrix} \in \mathbf{R}^{m\times p}.$$

- Column partition:

$$B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{pmatrix}, \quad \mathbf{b}_j \stackrel{def}{=} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{pmatrix} \in \mathbf{R}^m, \quad j = 1, \cdots, p.$$

- **Definition**:

$$A\,B = A\begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{pmatrix} \stackrel{def}{=} \begin{pmatrix} A\,\mathbf{b}_1 & A\,\mathbf{b}_2 & \cdots & A\,\mathbf{b}_p \end{pmatrix} \in \mathbf{R}^{n\times p}.$$

Matrix-matrix product

- Let

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \in \mathbf{R}^{n\times m}, \ B=\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{pmatrix} \in \mathbf{R}^{m\times p}.$$

- Column partition:

$$B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{pmatrix}, \quad \mathbf{b}_j \stackrel{def}{=} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{pmatrix} \in \mathbf{R}^m, \quad j = 1, \cdots, p.$$

- **Definition**:

$$A\,B = A \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{pmatrix} \stackrel{def}{=} \begin{pmatrix} A\,\mathbf{b}_1 & A\,\mathbf{b}_2 & \cdots & A\,\mathbf{b}_p \end{pmatrix} \in \mathbf{R}^{n\times p}.$$

- entry-wise formula:

$$(A\,B)_{jk} = (A\,\mathbf{b}_k)_j = \begin{pmatrix} a_{j1} & a_{j2} & \cdots & a_{jm} \end{pmatrix} \mathbf{b}_k = \sum_{i=1}^{m} a_{ji}\, b_{ik}.$$

Matrix-matrix product, example

- Let
$$A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \\ 1 & 4 \end{pmatrix} \in \mathbf{R}^{3\times 2}, \; B = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 1 & 2 \end{pmatrix} \in \mathbf{R}^{2\times 3}.$$
- $$C = AB = \begin{pmatrix} 12 & 5 & 1 \\ 1 & 0 & 3 \\ 14 & 5 & 7 \end{pmatrix} \in \mathbf{R}^{3\times 3}.$$