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Math128A: Numerical Analysis Sample Final Exam

This is a closed book, closed notes exam. You need to justify every one of your answers. Completely correct answers given without justification will receive little credit. Do as much as you can. Partial solutions will get partial credit. Look over the whole exam to find problems that you can do quickly.

Problem	Maximum Score	Your Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Your Name: ______________________________

Your SID: ______________________________

Your GSI: ______________________________

1. Consider a quadrature of the form

$$\int_{-1}^{1} |x| f(x) dx = \frac{1}{4} \left(f(-1) + 2f(0) + f(1) \right).$$

Show that it is exact for any polynomial $f(x)$ of degree at most 3.

2. Let $A \in \mathbf{R}^{n\times n}$ be a non-singular upper triangular banded matrix with band width $p > 0$. In other words, let $a_{i,j}$ be the (i,j) entry of A. Then $a_{i\,j} = 0$ if $i > j$ or $j - i > p$.

   (a) Sketch an efficient algorithm that solves the linear equations $Ax = b$ in $O(np)$ operations.

   (b) Count the number of operations (multiplications, additions, subtractions, divisions) of your algorithm up to the np term.

   (c) Sketch an efficient algorithm that solves the linear equations $AX = B$ in $O(n^2 p)$ operations, where both the unknown X and right hand side B are $n \times n$ matrices.

3. Determine the exact conditions on the coefficients a, b, c, d, e under which the following function is a cubic spline:

$$f(x) = \begin{cases} a(x-2)^2 + b(x-1)^3, & \text{if} \quad x \in (-\infty, 1], \\ c(x-2)^2, & \text{if} \quad x \in (1, 3], \\ d(x-3) + e(x-2)^2, & \text{if} \quad x \in [3, \infty). \end{cases}$$

4. Consider the Newton's method

$$x_{k+1} = x_k - f(x_k)/f'(x_k), \quad k = 1, 2, \cdots$$

for finding the root of the equation $f(x) = 0$. Assume that

- x^* is a multiple root, i.e., $f(x) = (x - x^*)^m g(x)$, where $m > 1$ is an integer and $g(x)$ is a smooth function with $g(x^*) \neq 0$.
- Newton's method converges to x^*.

Show that Newton's method must converge to x^* linearly.

5. - Consider the implicit Euler's method

$$w_{k+1} = w_k + hf\left(t_{k+1}, w_{k+1}\right)$$

   for $k = 0, 1, \cdots,$. Find its region of absolute stability.

   - Consider a multi-step method of the form

$$w_{k+1} = w_k + h\left(Bf\left(t_k, w_k\right) + Cf\left(t_{k-1}, w_{k-1}\right)\right),$$

   where $h > 0$ is the step-size and $t_k = kh$ for all k.

   (a) Define what is meant by the local truncation error for the multi-step method.

   (b) For $B = 3/2$ and $C = -1/2$, show that the local truncation error of this method is of second order.