Midterm Solutions

Problem 2:

(a) Our function is $f(x) = (3x+1)^{1/3}$. At $x = 1$, $f(x) - x = 4^{1/3} - 1$. This is positive, since $1^3 < 4$. At $x = 2$, $f(x) - 2 = 7^{1/3} - 2$. This is negative, since $2^3 > 7$. By the Intermediate Value Theorem, $f(x) - x$ has a root in the interval $[1, 2]$. Equivalently, $f(x)$ has a fixed point in this interval.

(b) Fixed point iteration for $f(x)$ converges in an interval $[a, b]$ if there is a root in this interval and $|f'(x)| < \lambda < 1$ for all $x \in [a, b]$. From part (a), we know there is a root. We compute $f'(x) = \frac{1}{3}3(3x+1)^{-2/3} = (3x+1)^{-2/3}$. Since $3x + 1 \geq 4$ for x in $[1, 2]$, we conclude that $(3x+1)^{-2/3} \leq 4^{-2/3} < 1$ for all x in the interval. This satisfies the conditions of the theorem, so the fixed point iteration $p_{k+1} = (3p_k + 1)^{1/3}$ convergers for any starting point $p_0 \in [1, 2]$.

Problem 3:

We want to find a polynomial $P(x)$ of degree at most 2 such that $P(0) = P(1) = P(2) = 1$. We compute the Lagrange polynomials $L_0 = \frac{(x-1)(x-2)}{(0-1)(0-2)}$, $L_1 = \frac{(x-0)(x-2)}{(1-0)(1-2)}$, $L_2 = \frac{(x-0)(x-1)}{(2-0)(2-1)}$. Our polynomial P should be $1L_0 + 1L_1 + 1L_2$ $= (x^2 - 3x + 2)/2 - (x^2 - 2x) + (x^2 - x)/2 = 0x^2 + 0x + 1$. So $P(x) = 1$.

We verify $P(1) = P(2) = P(3) = 1$, so this is the correct polynomial.

Problem 4:

Relative error is $\frac{|p-p^*|}{|p|}$, and this should be at most 10^{-4}. So $|p - p^*| \leq 10^{-4}p$, or $-10^{-4}p \leq p - p^* \leq 10^{-4}p$. This is true for p^* in the interval $[p - 10^{-4}p, p + 10^{-4}p]$. Since $p = \sqrt{2}$, this is $[\sqrt{2} - 10^{-4}\sqrt{2}, \sqrt{2} + 10^{-4}\sqrt{2}]$.

An estimate has n significant digits if the relative error is less than $5 \cdot 10^{-n}$. As we have just computed, for p in the above interval the relative error is at most $10^{-4} < 5 \cdot 10^{-4}$, so any p^* in this interval has at least 4 significant digits.

Problem 5:

(a) $p_k = \lambda^{\alpha^k}$, with $\lambda < 1$ and $\alpha > 0$. Since $\lambda < 1$, p_k approaches zero as $k \to \infty$ if and only if $\alpha^k \to \infty$. This is true if and only if $\alpha > 1$, which is our condition for p_k to approach zero.

(b) We compute the order of convergence as the largest R for which $\lim_{k\to\infty} \frac{p_{k+1}}{p_k^R}$ exists. This is $\frac{\lambda^{\alpha^{k+1}}}{(\lambda^{\alpha^k})^R}$, which is equal to $\frac{\lambda^{\alpha(\alpha^k)}}{\lambda^{R(\alpha^k)}} = \lambda^{(\alpha-R)\alpha^k}$. Since $\alpha > 1$ from part (a), this limit exists if and only if $R \leq \alpha$, so the order of convergence is α.