(2) The given method determines the integral from a to b by considering the value of the function at five points.

In order for it to be fifth order, it should give the exact solutions for polynomials of degree at most 4. We will consider polynomials of the form $(x-(a+2h))^k$ on the interval $(a,b)=(a,a+4h)$ to make the computations as simple as possible. Its integral is $(x-(a+2h))^{k+1}/(k+1)|_a^{a+4h} = ((2h)^{k+1} - (-2h)^{k+1})/(k+1)$.

$\int_a^b 1dx = 4h$.

$I_4(1) = \frac{2h}{45}(7+32+12+32+7) = \frac{2h}{45}(90) = 4h$.

$\int_a^b (x-(a+2h))dx = (2h)^2/2 - (-2h)^2/2 = 0$.

$I_4(x-a) = \frac{2h}{45}(7\cdot 2h - 32\cdot h - 12\cdot 0 + 32\cdot h + 7\cdot 2h) = 0$.

$\int_a^b (x-a-2h)^2 dx = (2h)^3/3 - (-2h)^3/3 = \frac{16}{3}h^3$

$I_4((x-(a+2h))^2) = \frac{2h}{45}(7\cdot 4h^2 + 32h^2 + 0 + 32\cdot h^2 + 7\cdot 16h^2) = \frac{2h^3}{45}(28 + 32 + 32 + 28) = \frac{2\cdot 120}{45}h^3 = \frac{16}{3}h^3$

$\int_a^b (x-a)^3 dx = (2h)^4/4 - (-2h)^4/4 = 0$

$I_4((x-(a+2h))^3) = \frac{2h}{45}(-7\cdot 8h^3 - 32\cdot h^3 + 0 + +32h^3 + 7\cdot 8h^3) = 0$.

$\int_a^b (x-(a+2h))^4 dx = (2h)^5/5 - (-2h)^5/5 = \frac{64}{5}h^5$

$I_5((x-(a+2h))^4) = \frac{2h}{45}(7\cdot 16h^4 + 32h^4 + 0 + 32h^4 + 7\cdot 16h^4) = \frac{2h^4}{45}(112 + 32 + 32 + 112) = \frac{2\cdot 288}{45} = \frac{64}{5}h^5$.

$\int_a^b (x-(a+2h)))^5 dx = 0$

$I_5((x-(a+2h))^5)$ is also zero; the first two terms cancel the last two as above.

So the above method gives the exact solution on any polynomial of the form $(x-(a+2h))^k$, $0 \le k \le 5$. Any polynomial of degree 5 or lower may be written as a linear combination of such polynomials, so the formula has degree of precision at least 5.

To see that it is not accurate to degree 6, we consider the polynomial $(x-(a+2h)^6)$. Its correct integral from a to b is $2(2h^7)/7$. But the formula gives an integer multiple of $h^7/45$, proceeding in the same way as above. This cannot be the correct answer since 7 does not divide 45.

(b) For the composite formula, we divide the interval $[a,b]$ into $4m$ subintervals, each of width $h = \frac{b-a}{4m}$. We then apply Boole's Rule to each set of four intervals, obtaining the formula $I(f) = \frac{2h}{45}\sum_{k=0}^{m-1}(7f(a+4kh) + 32f(a + 4kh + h) + 12f(a+4kh+2h) + 32f(a+4kh+3h) + 7f(a+4kh+4h))$.

(3) $f(x) = sin(x)/x$. We approximate by natural spline at $x = -1, 0, 1$. $f(-1) = sin(-1)/(-1) = sin(1)$, $f(1) = sin(1)/1 = sin(1)$. To find $f(0)$ we take the limit $lim_{x\to 0} sin(x)/x = cos(0)/1 = 1$ by L'Hopital's rule. So our points are $f(-1) = f(1) = sin(1)$, $f(0) = 1$.

Now we find the natural spline P. We have eight variables: on $[-1, 0]$ we have $P(x) = a(x+1)^3 + b(x+1)^2 + c(x+1) + d$, while on $[0, 1]$ we have $P(x) = qx^3 + rx^2 + sx + t$.

Since the spline must match the value of f at the given points, we have $P(-1) = d = sin(1)$, $P(0) = a+b+c+d = t = 1$, $P(1) = q+r+s+t = sin(1)$.

$P'(x)$ is $3a(x+1)^2 + 2b(x+1) + c$ on $[-1, 0]$ and $3qx^2 + 2rx + s$ on $[0, 1]$. For the first derivative to be continuous at 0, we must have $3a + 2b + c = s$.

$P''(x)$ is $6a(x+1) + 2b$ on $[-1, 0]$ and $6qx + 2r$ on $[0, 1]$. For the second derivative to be continuous at 0, we must have $6a + 2b = 2r$.

Finally, the natural spline condition is that the second derivative should be zero at the endpoints. This gives $2b = 0$, $6q + 2r = 0$.

Putting all this together we have

$d = sin(1)$, $t = 1$, $b = 0$

$a + c = 1 - sin(1)$

$q + r + s = sin(1) - 1$, so $a + c + q + r + s = 0$.

$3a + c = s$

$6a = 2r$, $a = r/3$

$6q + 2r = 0$, so $q = -r/3$.

Then $s = r + c$, $r/3 + c - r/3 + r + r + c = 0 = 2c + 2r$, and $c = -r$, implying $s = 0$.

We then have $r/3 - r = -2r/3 = 1 - sin(1)$, for $r = -\frac{3}{2}(1 - sin(1))$. So we can write down P as follows:

$P(x) = -\frac{1}{2}(1 - sin(1))(x+1)^3 + \frac{3}{2}(1 - sin(1))(x+1) + sin(1)$ on $[-1, 0]$ and

$P(x) = \frac{1}{2}(1 - sin(1))x^3 - \frac{3}{2}(1 - sin(1))x^2 + 1$ on $[0, 1]$.

(4) $L = lim_{h\to 0} f(h)$ and $L - f(h) = c_6 h^6 + c_9 h^9 + ...$

We consider $L - f(h/2)$. This is $c_6(h/2)^6 + c_9(h/2)^9... = c_6 h^6/64 + c_9 h^9/512$. To cancel the lowest order term, we compute $(L - f(h)) - 64(L - f(h/2)) = c_9 h^9 - (c_9/8)h^9$.

Then $-63L - f(h) + 64f(h/2) = (7/8)c_9 h^9$. Solving for L we find $L = (64f(h/2) - f(h) - (7/8)c_9 h^9)/63$, so $L - (\frac{64}{63}f(h/2) - \frac{1}{63}f(h)) = -\frac{1}{72}c_9 h^9$. This is order h^9 instead of h^6, and so is more accurate than the original estimate.

(5) We have $f(0) = 0$, $f(0.1) = 0.01$, $f(0.2) = 0.04$.

A second-order method must give the exact solution on quadratic polynomials. Therefore the second-order approximation to the derivative of f using these three points must be equal to the derivative of the quadratic passing through these points.

We compute this quadratic with Lagrange polynomials: $P(x) = 0L_0 + 0.01L_1 + 0.04L_2 = 0.01\frac{x(x-0.2)}{(0.1-0)(0.1-0.2)} + 0.04\frac{x(x-0.1)}{(0.2-0)(0.2-0.1)}$.

This is $P = -0.01\frac{x^2-0.2x}{0.01} + 0.04\frac{x^2-0.1x}{0.02}$, which is equal to $-(x^2 - 0.2x) + 2(x^2 - 0.1x) = x^2$. So our second-order approximation to $f'(x)$ at $x = 0$ is the derivative of x^2 at 0, which is 0.