Practice Midterm Solutions

Problem 2:

$f(x) = 2x^3 - 6x + 1$. $f(0) = 0 + 0 + 1 = 1$, $f(1) = 2 - 6 + 1 = -3$, so by the Intermediate Value Theorem f has a root in the interval $[0, 1]$.

Bisection method:

$f(1/2) = \frac{2}{8} - \frac{6}{2} + 1 = -\frac{7}{4}$. $f(1/2) < 0 < f(0)$. Therefore f has a root in $[0, 1/2]$.

$2x^3 - 6x + 1 = 0$ if and only if $x = 2x^3 - 5x + 1$. We look for a fixed point of $g(x) = 2x^3 - 5x + 1$. Let $p_0 = 1/2$. then $g(p_0) = \frac{2}{8} - \frac{5}{2} + 1 = -\frac{5}{4}$. So our next approximation to the fixed point is $p_1 = -\frac{5}{4}$.

Problem 3:

We want to find a second-degree polynomial P with $P(x_0) = f_0$, $P(x_1) = f_1$, and $P'(x_2) = f_2'$, where $x_0 < x_1 < x_2$.

Following the hint, we write $P(x) = \alpha + \beta(x - x_0) + \gamma(x - x_0)(x - x_1)$.

Then $P(x_0) = f_0 = \alpha + 0 + 0$, so $\alpha = f_0$.

We then have $P(x_1) = f_1 = f_0 + \beta(x_1 - x_0) + 0$, so $\beta = (f_1 - f_0)/(x_1 - x_0)$.

$P_2'(x) = \beta + \gamma((x - x_1) + (x - x_0))$ $(f_1 - f_0)/(x_1 - x_0) + \gamma(2x - x_0 - x_1)$.

Therefore $P'(x_2) = f_2' = (f_1 - f_0)/(x_1 - x_0) + \gamma(2x_2 - x_0 - x_1)$. This gives $\gamma = (f_2' - \frac{f_1 - f_0}{x_1 - x_0})/(2x_2 - x_0 - x_1)$.

For these values of α, β, γ all three equations are satisfied, so $P(x) = \alpha + \beta(x - x_0) + \gamma(x - x_0)(x - x_1)$ is the desired polynomial.

Problem 4:

We look for the unique positive root of the equation $f(x) = x^3 - a = 0$.

(a) If x_i is the ith approximation to the root, then Newton's method gives $x_{i+1} = x_i - f(p_i)/f'(p_i)$. This is $x_i - \frac{x^3 - a}{3x^2}$.

(b) $a = 2$, $x_0 = 1$.

$f(1) = -1$, $f'(1) = 3$.

$x_1 = 1 - \frac{-1}{3} = 4/3$.

$f(4/3) = \frac{64}{27} - 2 = \frac{64 - 54}{27} = \frac{10}{27}$.

$f'(4/3) = 3 \cdot \frac{16}{9} = \frac{16}{3}$.

$x_2 = 4/3 - \frac{10/27}{16/3} = \frac{576}{432} - \frac{30}{432} = \frac{546}{432} = \frac{91}{72}$.

(c) A convergent iteration p_i is of order n if p_i converges and $lim_{i \to \infty} |\frac{p_{i+1} - p_i}{(p_i - p_{i-1})^n}|$ exists.

$f'(2^{1/3}) = 3 \cdot 2^{2/3} \neq 0$, so the root $p = 2^{1/3}$ is a simple root. Newton's method converges to second order at any simple root, so it does so in this case.

Problem 5:

If p^* is some approximation to a quantity p, then the absolute error is $|p - p^*|$ and the relative error is $|p - p^*|/|p|$.

An estimate p^* has n significant digits if the relative error is less than or equal to $5 \cdot 10^{-n}$.