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Math128A: Numerical Analysis

Programming Assignments #3 and #4, Due Nov. 19, 2008

In classical mechanics, the two-body problem is to determine the motion of two point particles that interact only with each other. An example is the earth orbiting the sun.

Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the positions of the two bodies, the Earth, and the Sun, and let m_1 and m_2 be their masses. We would like to determine the trajectories $x_1(t)$ and $x_2(t)$ for time t, given initial positions $\mathbf{x}_1(t == 0)$, $\mathbf{x}_2(t == 0)$, and initial velocities $\mathbf{v}_1(t == 0)$, $\mathbf{v}_2(t == 0)$. According to Newton's second law,

$$F_{12}(\mathbf{x}_1, \mathbf{x}_2) = m_1 \ddot{\mathbf{x}}_1,$$
$$F_{21}(\mathbf{x}_1, \mathbf{x}_2) = m_2 \ddot{\mathbf{x}}_2,$$

where $F_{12}(\mathbf{x}_1, \mathbf{x}_2)$ is the force on mass 1 due to its interaction with mass 2; and $F_{21}(\mathbf{x}_1, \mathbf{x}_2) = -F_{12}(\mathbf{x}_1, \mathbf{x}_2)$ is the force on mass 2 due to its interaction with mass 1.

We assume that

- The two masses are on a plane, so that $\mathbf{x}_1$ and $\mathbf{x}_2$ have two components each.
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  $$F_{12}(\mathbf{x}_1, \mathbf{x}_2) = \frac{\gamma m_1 m_2}{r^\alpha} \left(\mathbf{x}_2 - \mathbf{x}_1 \right),$$
  where α is a given constant and $r = \|\mathbf{x}_2 - \mathbf{x}_1\|$.

We choose the following set of parameters:

- $m_1 = 10$, $m_2 = 15000$;
- $\gamma = 4 \times 10^3$;
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  $$\mathbf{x}_1(t == 0) = \begin{pmatrix} 1000 \\ 0 \end{pmatrix} \quad \mathbf{x}_2(t == 0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_1(t == 0) = \begin{pmatrix} 0 \\ 300 \end{pmatrix}, \quad \mathbf{v}_2(t == 0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

- tolerance $\tau = 1 \times 10^{-8}$;
- **hmin** $= 1 \times 10^{-5}$; **hmax** $= 1 \times 10^{-1}$;
- time interval on which to recover the orbits: $[0, 200]$.

Reformulate this problem as a system of first order ODEs.

For Programming Assignment #3:

- Solve the system of ODEs using both the matlab function `ode45` and the matlab code `RKFv.m` on the class website, for $\alpha = 2.95, 3, 3.05$. Only $\alpha = 3$ will give a closed orbit for the Earth.
- Your output should be three plots depicting the orbits of the Earth and the Sun for the three values of α from both methods. On the plots comment on the amount of time `ode45` and `RKFv` took to finish their jobs.

For Programming Assignment #4, we let $\alpha = 3$. Let $\mathbf{x}_1 = (u(t) \quad v(t))^T$. The calculations of `ode45` provides values of $u(t)$ and $v(t)$ at a selected set of points in time.

- Use clamped splines to generate a function that can compute approximations to $u(t)$ and $v(t)$ at any time t in $[0 \quad 200]$.
- Calculate the period T (the time it takes) for the Earth to complete one orbit (going from $\begin{pmatrix} 1000 \\ 0 \end{pmatrix}$ to approximately $\begin{pmatrix} 1000 \\ 0 \end{pmatrix}$).
- Calculate the distance D the Earth travels in one period:

$$D = \int_0^T \sqrt{(u'(t))^2 + (v'(t))^2}\, dt.$$

  The derivatives $u'(t)$ and $v'(t)$ can be obtained by differentiating the splines within each subinterval. Rewrite this integral as the sum of 4 subintegrals on intervals of length $T/4$ each, and compute each subintegral with the Gaussian quadrature of degree $n = 100$.
- Report the values of T and D on the orbit plot for $\alpha = 3$ from Assignment #3.