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Math128A: Numerical Analysis Final Exam

This is a closed book, closed notes exam. You need to justify every one of your answers. Completely correct answers given without justification will receive little credit. Do as much as you can. Partial solutions will get partial credit. Look over the whole exam to find problems that you can do quickly.

Problem	Maximum Score	Your Score
1	4	
2	16	
3	16	
4	16	
5	16	
6	16	
7	16	
Total	100	

1. (4 Points)

   Your Name: ______________________________

   Your SID: ______________________________

   Your GSI: ______________________________

2. (16 Points) Determine a polynomial $p(x)$ of degree at most 2 such that $p(-1) = 1$, $p(0) = 0$, and $p(1) = 1$.

3. (16 Points) Consider a multi-step method of the form

$$w_{k+1} = w_k + h\left(Bf\left(t_k, w_k\right) + Cf\left(t_{k-1}, w_{k-1}\right)\right),$$

where $t_k = kh$ for all k.

(a) Define what is meant by the local truncation error for the multi-step method.

(b) Choose the constants B and C so that this method has the highest order of local truncation error.

4. (16 Points)

   (a) Use the Simpson's Rule

   $$\int_a^b f(x)dx \approx \frac{b-a}{6}\left(f(a) + 4f(\frac{a+b}{2}) + f(b)\right)$$

   to evaluate the integral

   $$\int_{-1}^{1} |x|x^2 dx$$

   .

   (b) Consider a quadrature of the form

   $$\int_{-1}^{1} |x|\, f(x)dx = A\,(f(x_1) + f(x_2))\,.$$

   Determine the constants A, x_1 and x_2 so quadrature has the highest degree of precision with respect to function $f(x)$. What is the degree of precision?

5. (16 Points)

   (a) Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$. Compute the LU factorization of A without pivoting.

   (b) Let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and assume $a \neq 0$. Let $A = L\,U$ be the LU factorization of A without pivoting. Show that the pivots of A are positive if and only if A is symmetric positive definite. Recall that A is symmetric positive definite if and only if $x^T A x > 0$ for any non-zero 2-dimensional vector x.

6. (16 Points)

   (a) Let A and B be $n \times n$ matrices. Prove or find a counter example: If $\det(A+B) = 0$ then $\det(A) = 0$ or $\det(B) = 0$.

   (b) Let A be an $n \times n$ matrix. The matrix 2-norm and **max**-norm are defined as

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \quad \text{and} \quad \|A\|_{\mathbf{max}} = \max_{1 \leq i,j \leq n} |a_{i,j}|.$$

   Show that

$$\|A\|_{\mathbf{max}} \leq \|A\|_2 \leq n\|A\|_{\mathbf{max}}.$$

7. (16 Points) Consider the fixed point iteration

$$x_{k+1} = (\alpha + 1)x_k - x_k^2, \quad k = 0, 1, \cdots,$$

where α satisfying $1 \geq \alpha \geq \frac{1}{2}$ is given.

(a) Show that the iteration converges for any initial guess x_0 satisfying $\alpha - \frac{1}{5} \leq x_0 \leq \alpha + \frac{1}{5}$.

(b) Assume that the iteration converges, find the value of α for which the method converges quadratically.