MIXED NORM ESTIMATES FOR CERTAIN GENERALIZED RADON TRANSFORMS

MICHAEL CHRIST AND M. BURAK ERDOĞAN

1. Introduction

In this paper we investigate the mapping properties in Lebesgue-type spaces of certain generalized Radon transforms defined by integration over curves.

Let X and Y be open subsets of $\mathbf{R}^d$, $d \geq 2$, and let Z be a smooth submanifold of $X \times Y \subset \mathbf{R}^{2d}$ of dimension $d+1$. Assume that the projections $\pi_1 : Z \to X$ and $\pi_2 : Z \to Y$ are submersions at each point of Z. For each $y \in Y$, let

$$\gamma_y = \{x \in X : (x, y) \in Z\} = \pi_1 \pi_2^{-1}(y).$$

In this case, γ_y are smooth curves in X which vary smoothly with $y \in Y$. For every $y \in Y$, choose a smooth, non-negative measure σ_y on γ_y which varies smoothly with y in the natural sense. A generalized Radon transform T (see e.g. [2, 10, 12]) is defined as an operator taking functions on X to functions on Y via

$$Tf(y) = \int_{\gamma_y} f d\sigma_y.$$

The adjoint of this operator has a similar form:

$$T^* g(x) = \int_{\gamma_x^*} g d\sigma_x^*,$$

where

$$\gamma_x^* = \{y : (x, y) \in Z\} = \pi_2 \pi_1^{-1}(\{x\}) \subset Y$$

and σ_x^* is a nonnegative measure on γ_x^* with a smooth density which varies smoothly with x.

Tao and Wright [13] have formulated and proved a nearly optimal characterization of the local (L^p, L^q) mapping properties of these operators. We extend their result to the mixed-norm setting and obtain essentially optimal local mixed-norm inequalities for these operators, under one additional dimensional restriction. Previously this result was obtained for a model operator in [14, 6, 4]. See [1, 4, 6, 13, 14] for various examples and prior work, and [2] for a partially alternative development of the unmixed norm theory.

Date: 9/6/2005. Revised 11/7/2006.

The first and second authors were partially supported by NSF grants DMS-040126 and DMS-0540084, respectively.

Mixed norms on Y. Throughout the entire discussion, X, Y will denote sufficiently small neighborhoods of x_0, y_0 for some fixed point $(x_0, y_0) \in Z$. Let $\Pi : Y \to \mathbf{R}$ be a submersion such that the fibers $\Pi^{-1}(t)$ are transverse to the curves γ_x^*. This means that the restriction of Π to γ_x^* is a diffeomorphism for each $x \in X$. Choose coordinates so that $\Pi(y_0) = 0$. Let λ_t be Lebesgue measure on the $d-1$-dimensional surface $\Pi^{-1}(t)$. To a function $f : Y \mapsto \mathbf{R}$ we associate the mixed norms

$$\|f\|_{L^qL^r(Y)} = \|f\|_{q,r} := \left[\int_{\mathbf{R}}\left[\int_{\Pi^{-1}(t)} |f(s)|^r \, d\lambda_t(s)\right]^{q/r} dt\right]^{1/q}.$$

The integral with respect to t is taken over a small neighborhood of the origin in $\mathbf{R}$; we may assume that this neighborhood is contained in $[-1, 1]$. Here, and throughout this paper, t is restricted to lie in a one-dimensional manifold. This is *not* a natural restriction, but our analysis yields reasonably satisfactory results only in this special case.

This situation is quite general, and is satisfied for generic choices of the jets, up to a certain order depending only on d, of X, Y, Π at (x_0, y_0). Begin with any manifold Z of dimension $d + 1$, and a point $z_0 \in Z$. In a neighborhood of z_0 construct two real vector fields V_X, V_Y which are linearly independent at z_0, and which generate a Lie algebra that spans the tangent space to Z at z_0. Such vector fields exist in any dimension, and hence for any d there exists N such that any pair (V_X, V_Y) with generic jet up to order N at z_0 will satisfy this condition. Choose two codimension one manifolds $X, Y \subset Z$ in a neighborhood of z_0, such that the integral curves of V_X, V_Y passing through z_0 are transverse to X, Y, respectively, at z_0. There are natural submersions π_X, π_Y from Z to X, Y, defined by flowing from $z \in Z$ to X, Y along the integral curves of V_X, V_Y, respectively. The only requirement on $\Pi : Y \to \mathbf{R}$ is then that $V_X(\pi_Y \circ \Pi)(z_0) \neq 0$. Equivalently, $D\Pi(z_0)$ is required not to annihilate the push-forward under π_Y of $V_X(z_0)$. This is a single linear condition on $D\Pi(z_0)$, so any generic map Π satisfies the hypothesis. In particular, one can consider such mixed norms for each of the concrete examples discussed in [13].

A fundamental example which motivates our investigation has $X = \mathbf{R}^d$, Y equal to the manifold of all lines in $\mathbf{R}^d$, and $Z = \{(x, L) \in X \times Y : x \in L\}$. To any line L is associated $\Pi(L)$, the unique line parallel to L containing the origin. Π maps Y to the manifold G of all one-dimensional subspaces of $\mathbf{R}^d$. The x-ray transform maps $f : \mathbf{R}^d \to \mathbf{C}$ to $Tf : Y \to \mathbf{C}$, with $Tf(L)$ equal to the integral of f over L. Y can naturally be parametrized as the set of all $(\theta, x) \in G \times \mathbf{R}^{d-1}$ such that $x \perp \theta$; $L = x + \theta$ is a line parallel to θ. Thus one has mixed norms $L^q_\theta L^r_x$ for Y. Let B be a ball of finite radius in $\mathbf{R}^d$. A strong form of conjectured bounds for the Kakeya maximal function would say that T is bounded from $L^q(B)$ to $L^q_\theta L^r_x(G \times \mathbf{R}^{d-1})$, for all $r < \infty$ and $q > d$. The inequality holds for $d = 2$, and is unknown for $d \geq 3$. This example fails to satisfy our hypotheses in two respects for $d \geq 3$: Firstly, Y

has dimension $2d-2>d$. This means that the operator formally adjoint to T involves integration over submanifolds of dimension $d-1>1$, rather than over curves. Secondly, G has dimension strictly greater than one. In our view, this example amply motivates a general investigation of mixed-norm inequalities for generalized Radon transforms, even though such a general framework neglects special features of the motivating problem.

We say that T is of strong mixed type (p,q,r) if T maps $L^p(X)$ to $L^qL^r(Y)$ boundedly. We are mainly interested in local estimates. We assume throughout the discussion that T is L^p-improving, which means that for each $p\in(1,\infty)$, there exists $q>p$ such that T maps $L^p(X)$ to $L^q(Y)$. See [5] and [12] for characterizations of this property.

Our theorem is a characterization of the exponents (p,q,r) for which T is bounded. Before proceeding, let us record some simple facts about these exponents.

(i) Because of the transversality hypothesis described above, T is of strong mixed type (p,∞,p) for all $p\in[1,\infty]$.

(ii) Since we are working in a bounded region, whenever T is of strong mixed type (p,q,r), it is also of strong mixed type (p_1,q_1,r_1) whenever $p_1\geq p$, $q_1\leq q$, and $r_1\leq r$.

(iii) Because of (i) and (ii), T is of strong mixed type (p,q,r) whenever $p\geq r$.

Two-parameter Carnot-Carathéodory Balls. Tao and Wright [13] related the set of all exponents (p,q) for which the operator T maps $L^p(X)$ to $L^q(Y)$ to the geometry of Z. To describe this relation, choose smooth nowhere-vanishing linearly independent real vector fields V_1, V_2 on Z whose integral curves are the fibers of π_1, π_2 respectively. Equivalently, at each point $z\in Z$, V_j spans the nullspace of $D\pi_j$, for $j=1,2$. The L^p-improving property is equivalent to V_1,V_2 satisfying the bracket condition, i.e., V_1 and V_2 together with their iterated commutators span the tangent space to Z at each point in Z [5].

Definition 1. *Let $z_0\in Z$ and $0<\delta_1,\delta_2\ll 1$. The two-parameter Carnot-Carathéodory ball $B(z_0,\delta_1,\delta_2)$ consists of all the points $z\in Z$ such that there exists an absolutely continuous function $\varphi:[0,1]\to Z$ satisfying*
(i) $\varphi(0)=z_0$, $\varphi(1)=z$
(ii) for almost every $t\in[0,1]$

$$\varphi'(t)=a_1(t)V_1(\varphi(t))+a_2(t)V_2(\varphi(t))$$

with $|a_1(t)|<\delta_1$, $|a_2(t)|<\delta_2$.

The metric properties of Carnot-Carathéodory balls were studied extensively in [11]. The discussion there is phrased in terms of the one-parameter family of balls naturally associated to a family of vector fields satisfying the bracket condition. These balls depend on a center point, a radius r, and a family $\{W_j\}$ of vector fields. They can equivalently be viewed as depending on a center point and the family $\{rW_j\}$ of vector fields, with the radius redefined to be identically one. In these terms, the proofs in [11] go through more generally, for balls of radius one assigned to families of vector fields $\{U_j^\alpha : 1 \leq j \leq J\}$ satisfying the bracket condition for each parameter α, with appropriate uniformity as α varies. In particular, for the vector fields $\{\delta_1 V_1, \delta_2 V_2\}$, provided that $0 < \delta_1, \delta_2 \leq c_0$ for sufficiently small c_0, the conclusions of [11] hold uniformly in δ_1, δ_2 under a supplementary hypothesis of weak comparability, which is discussed below. See [13, 2].

It will be convenient in our proof to parametrize the curves γ_x^* by t so that $\Pi(\gamma_x^*(t)) \equiv t$. With this parametrization, the measure σ_x on γ_x^* is equivalent to dt, uniformly in x. We rescale V_1 if necessary so that for each $z \in Z$ and sufficiently small $s \in \mathbf{R}$

$$\Pi\pi_2(e^{sV_1}z) = \Pi\pi_2(z) + s. \tag{1}$$

Definition 2. *Let $0 < \theta \leq 1$, and let A be a positive constant. We say that $0 < \delta_1, \delta_2 \ll 1$ are (θ, A)-weakly comparable, and write $\delta_1 \sim_{\theta,A} \delta_2$, if $\delta_1 \leq A\delta_2^\theta$ and $\delta_2 \leq A\delta_1^\theta$.*

The following lemma collects basic facts about the balls $B(z, \delta_1, \delta_2)$.

Lemma 1. *Let K be a compact subset of Z. Assume that $\delta_1 \sim_{(\theta,A)} \delta_2$ are sufficiently small, and let $z \in K$. Then $B = B(z, \delta_1, \delta_2)$ satisfies*
(i) $|B| \sim |B(z, 2\delta_1, 2\delta_2)|$,
(ii) $|B| \sim |\pi_1(B)|\delta_1 \sim |\pi_2(B)|\delta_2$
(iii) $|\Pi(\pi_2(B))| \sim \delta_1$,
(iv) $\|\chi_{\pi_2(B)}\|_{q',r'} \sim |B|^{1-\frac{1}{r}}\delta_1^{\frac{1}{r}-\frac{1}{q}}\delta_2^{\frac{1}{r}-1}$,
(v)

$$\frac{|B|}{|\pi_1(B)|^{\frac{1}{p}}\|\chi_{\pi_2(B)}\|_{q',r'}} \sim |B|^{\frac{1}{r}-\frac{1}{p}}\delta_1^{\frac{1}{p}+\frac{1}{q}-\frac{1}{r}}\delta_2^{1-\frac{1}{r}}.$$

Here $1 \leq p, q, r \leq \infty$, and q', r' are the exponents conjugate to q, r, respectively.

The notation $A \sim C$ means that the ratio A/C is bounded above and below by quantities depending on Z, θ, A, and the compact set K, but not on δ_1, δ_2. In the absence of weak comparability, the doubling property (i) fails in general for two-parameter Carnot-Carathéodory balls associated to C^∞ vector fields satisfying the bracket condition [2].

For a sketch of the proof of the lemma see §4 below.

Statement of results. Recall that T is said to be of restricted weak type (p, q) if for all Lebesgue measurable sets $E \subset X$ and $F \subset Y$

$$\langle T\chi_E, \chi_F\rangle \lesssim |E|^{1/p}|F|^{1/q'}, \tag{2}$$

where q' denotes the exponent conjugate to q. In our setup

$$\langle T\chi_E, \chi_F\rangle \approx |\pi_1^{-1}(E) \cap \pi_2^{-1}(F)|, \tag{3}$$

where $|\cdot|$ denotes Lebesgue measure on Z. We test the inequality (2) on the Carnot-Carathéodory balls $B(z, \delta_1, \delta_2)$ under the restriction that $\delta_1 \sim_{(\theta,A)}$

δ_2. Let $E = \pi_1(B(z, \delta_1, \delta_2)), F = \pi_2(B(z, \delta_1, \delta_2))$. Using (3), Lemma 1 and restricting attention to the nontrivial case where $q > p$, the inequality (2) reads

$$|B(z, \delta_1, \delta_2)| \gtrsim \delta_1^{c_1} \delta_2^{c_2}, \tag{4}$$

where

$$c_1 = \frac{p^{-1}}{p^{-1} - q^{-1}}, \qquad c_2 = \frac{1 - q^{-1}}{p^{-1} - q^{-1}}. \tag{5}$$

Define

$$\mathcal{C}_{\theta,A}(T) := \{(c_1, c_2) : \inf \left(\frac{|B(z, \delta_1, \delta_2)|}{\delta_1^{c_1} \delta_2^{c_2}} \right) > 0\},$$

where the infimum is taken over all $z \in Z$ and over all pairs δ_1, δ_2 that satisfy $\delta_1 \sim_{(\theta,A)} \delta_2$. Define

$$\mathcal{C}(T) := \cap_{0<\theta\leq 1} \cap_{A\geq 1} \mathcal{C}_{\theta,A}(T).$$

According to (4), (2) can not hold for (p, q) if the corresponding (c_1, c_2) does not belong to $\mathcal{C}(T)$. Tao and Wright [13] proved that[1] for all (c_1, c_2) in the interior of $\mathcal{C}(T)$, (2) holds for the exponents (p, q) defined by (4).

In this note we extend this result to mixed norms. We say that T is of restricted weak mixed type (p, q, r) if for all $E \subset X$ and $F \subset Y$,

$$\langle T\chi_E, \chi_F \rangle \lesssim |E|^{1/p} \|\chi_F\|_{q',r'}.$$

By interpolation, the strong mixed type estimates can be obtained from these inequalities, except for exponents corresponding to boundary points of $\mathcal{C}(T)$.

The two-parameter Carnot-Carathéodory balls defined above also dictate the allowed exponent triples (p, q, r) for mixed norm inequalities, under certain additional restrictions on the exponents p, q, r:

Definition 3. *Let $P_{\theta,A}(T)$ be the set of all exponents (p, q, r) satisfying*
(i) $1 \leq p \leq q \leq r \leq \infty$,
(ii)

$$\sup_{z,\delta_1,\delta_2} \frac{|B(z, \delta_1, \delta_2)|}{|\pi_1(B(z, \delta_1, \delta_2))|^{1/p} \|\chi_{\pi_2(B(z,\delta_1,\delta_2))}\|_{q',r'}} < \infty, \tag{6}$$

where q', r' are the conjugates of q, r respectively, and the supremum is taken over all $z \in Z$ and $\delta_1 \sim_{\theta,A} \delta_2$.

Using Lemma 1 we can rewrite the second condition in the definition of $P_{\theta,A}(T)$ as in (4) with

$$c_1 = \frac{p^{-1} + q^{-1} - r^{-1}}{p^{-1} - r^{-1}}, \qquad c_2 = \frac{1 - r^{-1}}{p^{-1} - r^{-1}}. \tag{7}$$

[1] Tao and Wright defined the set $\mathcal{C}(T)$ differently. An analysis of the two-parameter balls along the lines of [11] establishes the equivalence of these two definitions.

Define

$$P(T) := \cap_{0<\theta\le 1} \cap_{A\ge 1} P_{\theta,A}(T) = \{(p,q,r) : r \ge q \ge p, (c_1,c_2) \in \mathcal{C}(T)\}.$$

We assume always that $r > p$, since otherwise the desired inequality holds automatically so long as π_1, π_2 are submersions, as discussed above.

It is natural to conjecture that if $1 \le p \le q \le r \le \infty$, then T is of restricted weak mixed type (p,q,r) if and only if $(p,q,r) \in P(T)$; we prove this conjecture except at the endpoints. In an appendix we explain the presence of the additional restriction $p \le q \le r$.

Theorem 1. *Let T be a generalized Radon transform of the class described above, and let $1 \le p \le q \le r \le \infty$. If (p,q,r) is in the interior of $P(T)$, then T maps $L^p(X)$ to $L^qL^r(Y)$ boundedly. Moreover, if $(p,q,r) \notin P(T)$, then T does not map $L^p(X)$ to $L^qL^r(Y)$ boundedly.*

2. A brief account of [13].

Given $E \subset X$ and , $F \subset Y$, let

$$\alpha_1 = \frac{\langle T\chi_E, \chi_F\rangle}{|E|}, \quad \alpha_2 = \frac{\langle T\chi_E, \chi_F\rangle}{|F|}. \tag{8}$$

(2) follows from an estimate of the form

$$|\Omega| := |\pi_1^{-1}(E) \cap \pi_2^{-1}(F)| \approx \langle T\chi_E, \chi_F\rangle \gtrsim \alpha_1^{c_1}\alpha_2^{c_2},$$

where c_1, c_2 are as in (5). Moreover, a loss of an arbitrarily small power of $\alpha_1\alpha_2$ is of no consequence since (c_1, c_2) belongs to the *interior* of $\mathcal{C}(T)$. Throughout the remainder of the discussion we will denote

$$\Omega := \pi_1^{-1}(E) \cap \pi_2^{-1}(F). \tag{9}$$

Let $\varepsilon > 0$ be a small exponent to be specified near the end of the proof, and let C_ε be a large constant. All constants in the ensuing discussion depend on X, Y, Z, c_1, c_2 and the quantities given as subscripts.

Definition 4. *Let $S \subset [-1,1]$ be a measurable set of positive Lebesgue measure. We say that S is central with width $w > 0$ if*
(i) $S \subset [-C_\varepsilon w, C_\varepsilon w]$ and
(ii) $|I \cap S| \le C_\varepsilon (|I|/w)^\varepsilon |S|$ for all intervals I.

Let Ω be a Lebesgue measurable subset of Z, having positive measure. We will assume throughout the discussion that $\pi_j(\Omega)$ has positive measure, for both $j = 1, 2$. Define

$$\alpha_j = \frac{|\Omega|}{|\pi_j(\Omega)|} \text{ for } j = 1,2 \tag{10}$$

$$\alpha := \min(\alpha_1, \alpha_2). \tag{11}$$

Fix any $\rho > 0$; ρ will eventually be chosen to be arbitrarily small at the conclusion of the proof.

Definition 5. *Let $\Omega \subset Z$ and let α_1, α_2 be as defined in* (10)*. Let j be an integer. A j-subsheaf Ω' of Ω, of width w_j, is a subset of Ω of measure $\geq C_\rho^{-1}\alpha^\rho|\Omega|$ such that for all $x \in \Omega'$, the set*

$$\{|t| \ll 1 : e^{tV_j}(x) \in \Omega'\}$$

is a central set of width w_j and measure $\geq C_{N,\varepsilon}\alpha^{C_N\varepsilon + C/N}\alpha_j$. Here $V_j = V_1$ of j is odd, and $V_j = V_2$ if j is even.

In [13] it is proved (see Corollary 8.3) that for any $\rho \in (0, 1]$ there exists $C_\rho < \infty$ such that for any measurable set $\Omega \subset Z$ there exists a nested sequence of subsets of Ω

$$\Omega_0 \subset \Omega_1 \subset ... \subset \Omega_{d+1} \subset \Omega$$

such that: For each j, Ω_j is a j-sheaf of Ω with width w_j,

$$C_\rho^{-1}\alpha^\rho\delta_j \leq w_j \leq \delta_j,$$

where δ_j is a 2-periodic sequence (that is, $\delta_{j+2} = \delta_j$ for all $0 \leq j \leq d-1$) with the properties

$$C_\rho^{-1}\alpha^\rho\alpha_j \leq \delta_j \leq 1,$$

and

$$\delta_1 \leq C_\rho\delta_2^\rho, \qquad \delta_2 \leq C_\rho\delta_1^\rho.$$

In particular, δ_1, δ_2 are weakly comparable, even though α_1, α_2 need not be; this ultimately explains why only balls with weakly comparable radii δ_1, δ_2 need be taken into account in the hypothesis of our theorem.

Using this construction, it is proved [13] that there exists some ball $B(z, \delta_1, \delta_2)$ such that

$$|B(z, \delta_1, \delta_2) \cap \Omega| \geq c\alpha^\varrho \left(\frac{\alpha_1}{\delta_1}\right)^{\lfloor (d+2)/2 \rfloor} \left(\frac{\alpha_2}{\delta_2}\right)^{\lfloor (d+1)/2 \rfloor} |B(z, \delta_1, \delta_2)|, \tag{12}$$

where $\varrho > 0$ can be made arbitrarily small by choosing ρ sufficiently small, and where $c > 0$ depends on ϱ but not on $\Omega, \alpha_j, \delta_j$. This implies that for arbitrarily small $\varrho > 0$

$$|\Omega| \gtrsim C_\varrho\alpha^\varrho\delta_1^{c_1}\delta_2^{c_2} \left(\frac{\alpha_1}{\delta_1}\right)^{\lfloor (d+2)/2 \rfloor} \left(\frac{\alpha_2}{\delta_2}\right)^{\lfloor (d+1)/2 \rfloor}, \tag{13}$$

for all (c_1, c_2) in the interior of $\mathcal{C}(T)$. This finishes the proof in the non mixed-norm case since[2] $c_1, c_2 \geq d \geq \lfloor (d+2)/2 \rfloor, \lfloor (d+1)/2 \rfloor$. The roles of X and Y are interchangeable. Therefore there is the alternative bound

$$|\Omega| \gtrsim C_\varrho\alpha^\varrho\delta_1^{c_1}\delta_2^{c_2} \left(\frac{\alpha_1}{\delta_1}\right)^{\lfloor (d+1)/2 \rfloor} \left(\frac{\alpha_2}{\delta_2}\right)^{\lfloor (d+2)/2 \rfloor}, \tag{14}$$

for all (c_1, c_2) in the interior of $\mathcal{C}(T)$.

[2]The bounds $c_1, c_2 \geq d$ are a simple consequence of an equivalent definition of $\mathcal{C}(T)$ given in [13].

3. Proof of Theorem 1.

The second conclusion of the theorem is immediate. If $(p,q,r) \notin P(T)$, then $(p,q,r) \notin P_{\theta,A}(T)$ for some θ, A. Therefore T can not be of mixed type (p,q,r).

For the first conclusion, it suffices to prove that for all (p,q,r) in the interior of $P(T)$ and for all $\eta > 0$, $\beta > 0$, and $F \subset Y$,

$$\langle T\chi_E, \chi_F\rangle \le C_{p,q,r,\eta}\beta^{-\eta}|E|^{1/p}\|\chi_F\|_{q',r'}, \tag{15}$$

where $E := \{x \in X : \beta < T^*\chi_F(x) \le 2\beta\}$.

Let $\Omega := \pi_1^{-1}(E) \cap \pi_2^{-1}(F)$. Note that

$$|\Omega| \approx \langle T\chi_E, \chi_F\rangle = \langle \chi_E, T^*\chi_F\rangle \approx \beta|E|. \tag{16}$$

For each $x \in E$, let $\mathcal{F}(x) := \{t \in \mathbf{R} : \gamma_x^*(t) \in F\} \subset \Pi(F) \subset [-1,1]$. Note that $|\mathcal{F}(x)| \approx \beta$.

Let $\eta > 0$ be a small constant to be specified at the end of the proof. Fix a small constant $c_\eta > 0$. Let $I(x) \subset [-1,1]$ be a dyadic interval of minimal length so that

$$|I(x) \cap \mathcal{F}(x)| \ge c_\eta |I(x)|^\eta |\mathcal{F}(x)|; \tag{17}$$

we choose c_η to guarantee that either $I = [-1,0]$ or $I = [0,1]$ satisfies (17). No interval of length $< (c_\eta|\mathcal{F}(x)|)^{1/(1-\eta)}$ can satisfy (17), so there must exist at least one dyadic interval of minimal length among all those satisfying the inequality.

This implies that for any dyadic subinterval $J \subset I(x)$,

$$|J \cap \mathcal{F}(x)| \le \frac{|J|^\eta}{|I|^\eta}|I(x) \cap \mathcal{F}(x)|. \tag{18}$$

Let

$$E_{m,k} = \{x \in E : |I(x)| \approx 2^m\beta, |I(x) \cap \mathcal{F}(x)| \approx 2^k\}.$$

Note that $\beta^{\eta/(1-\eta)} \lesssim 2^m \lesssim \beta^{-1}$ and $\beta^{1/(1-\eta)} \lesssim 2^k \lesssim \beta$.

By the pigeonhole principle there exists a pair m, k such that $\tilde{E} = E_{m,k}$ satisfies

$$\langle T\chi_E, \chi_F\rangle \le C_\eta \beta^{-\eta}\langle T\chi_{\tilde{E}}, \chi_F\rangle. \tag{19}$$

Choose one such pair, with which we work exclusively henceforth.

Partition $\mathbf{R}$ into intervals I_n, each of length $C2^m\beta$ where C is a sufficiently large fixed constant. Set

$$E_n = \{x \in \tilde{E} : I(x) \cap I_n \ne \emptyset\},$$

$$F_n = F \cap \Pi^{-1}(I_{n-1} \cup I_n \cup I_{n+1}).$$

Note that

$$\langle T\chi_{\tilde{E}}, \chi_F\rangle \lesssim \sum_n \langle T\chi_{E_n}, \chi_{F_n}\rangle. \tag{20}$$

Let $\Omega^n := \pi_1^{-1}(E_n) \cap \pi_2^{-1}(F_n)$. Then

$$2^k|E_n| \lesssim |\Omega^n| \leq \beta|E_n| \tag{21}$$

by Fubini's theorem. For on one hand, $|\pi_1^{-1}(x)| \leq \beta$ for all $x \in E$. On the other hand, $2^k \sim |I(x) \cap \mathcal{F}(x)|$ and for $x \in E_n$, $I(x) \cap \mathcal{F}(x) \subset (I_{n-1} \cup I_n \cup I_{n+1}) \cap \mathcal{F}(x) \subset \pi_2^{-1}(F_n)$. Using (18) with $I = I_{n-1} \cup I_n \cup I_{n+1}$, for each $x \in E_n$ and any dyadic interval J, we obtain

$$|J \cap \mathcal{F}(x)| \lesssim |J|^\eta (2^m\beta)^{-\eta} 2^k. \tag{22}$$

Let $\alpha_{n,i} = |\Omega^n|/|\pi_i(\Omega^n)|$, $i = 1, 2$. Then

$$2^k \lesssim \alpha_{n,1} \lesssim \beta, \tag{23}$$

by (21). Let $\alpha_n = \min(\alpha_{n,1}, \alpha_{n,2})$.

The following lemma follows from a simple application of Hölder's inequality; see [4].

Lemma 2. *Let $F \subset Y$. For $r \geq q$,*

$$\|\chi_F\|_{q'r'} \geq |F|^{1/r'} |\Pi(F)|^{1/q'-1/r'}.$$

We aim to prove that $\langle T\chi_{E_n}, \chi_{F_n}\rangle \lesssim |E_n|^{1/p} \|\chi_{F_n}\|_{q',r'}$. Via the preceding lemma, this would follow from

$$|\Omega^n| \approx \langle T\chi_{E_n}, \chi_{F_n}\rangle \gtrsim \alpha_{n,1}^{\gamma_1} \alpha_{n,2}^{\gamma_2} |\Pi(F_n)|^{\gamma_3}, \tag{24}$$

where

$$\gamma_1 = \frac{p^{-1}}{p^{-1} - r^{-1}}, \qquad \gamma_2 = \frac{1 - r^{-1}}{p^{-1} - r^{-1}}, \qquad \gamma_3 = \frac{q^{-1} - r^{-1}}{p^{-1} - r^{-1}}. \tag{25}$$

Whereas the exponent $\frac{1}{q'} - \frac{1}{r'}$ in Lemma 2 was nonpositive, here all the exponents γ_j are nonnegative.

The construction of Tao and Wright [13] produces a nested sequence of subsets $\Omega_0^n \subset \Omega_1^n \subset ... \subset \Omega_{d+1}^n \subset \Omega^n$ satisfying: For each j, Ω_j^n is a j-sheaf of Ω^n with width w_j,

$$C_\rho^{-1} \alpha_n^\rho \delta_j \leq w_j \leq C_\rho \delta_j,$$

where δ_j is a 2-periodic sequence with the property

$$C_\rho^{-1} \alpha_n^\rho \alpha_{n,j} \leq \delta_j \leq 1, \tag{26}$$

and $|\Omega_j^n| \geq c_\rho \alpha_n^\rho |\Omega^n|$. Moreover, δ_1, δ_2 are weakly comparable.

Now, we prove that the construction above guarantees a lower bound for δ_1. Indeed, let j be odd. Using (22), we have

$$|\Omega_j^n| \leq |E_n| \sup_{J,x} |J \cap F_n(x)| \leq |E_n| w_j^\eta (2^m\beta)^{-\eta} 2^k, \tag{27}$$

where the supremum is taken over all intervals J of length w_j and all points $x \in E_n$. Using (27), (21) and the fact that Ω_j^n is a j-sheaf of Ω^n, we get

$$c_\rho \alpha_n^\rho 2^k |E_n| \lesssim C_\rho \alpha_n^\rho |\Omega^n| \leq |\Omega_j^n| \leq |E_n| w_j^\eta (2^m\beta)^{-\eta} 2^k. \tag{28}$$

This implies that for odd j,

$$\delta_1 \gtrsim w_j \gtrsim 2^m \beta C_\rho^{1/\eta} \alpha_n^{\rho/\eta} \gtrsim |\Pi(F_n)| C_\rho^{1/\eta} \alpha_n^{\rho/\eta}. \tag{29}$$

Using the fundamental information (14), we obtain

$$|\Omega^n| \gtrsim \alpha_n^{\varrho} \alpha_n^{C\rho/\eta} \delta_1^{c_1} \delta_2^{c_2} \left(\frac{\alpha_{n,1}}{\delta_1}\right)^{\lfloor (d+1)/2 \rfloor} \left(\frac{\alpha_{n,2}}{\delta_2}\right)^{\lfloor (d+2)/2 \rfloor},$$

for any (c_1, c_2) belonging to the interior of $\mathcal{C}(T)$. Note that for $d \geq 2$ and for such (c_1, c_2), we have $0 \leq \gamma_3 < 1 \leq c_1 - \lfloor (d+1)/2 \rfloor$ and $c_2 > \lfloor (d+2)/2 \rfloor$. Using this, (26) and (29), we obtain

$$|\Omega^n| \gtrsim \alpha_n^{\varrho} \alpha_n^{C\rho/\eta} \delta_1^{\gamma_3} \alpha_{n,1}^{c_1 - \gamma_3} \alpha_{n,2}^{c_2} \gtrsim \alpha_n^{\varrho} \alpha_n^{C\rho/\eta} |\Pi(F_n)|^{\gamma_3} \alpha_{n,1}^{c_1 - \gamma_3} \alpha_{n,2}^{c_2},$$

for all (c_1, c_2) in the interior of $\mathcal{C}(T)$.

Given exponents $(p, q, r) \in P(T)$, define $\gamma_1, \gamma_2, \gamma_3$ by (25). Choose $\rho > 0$, depending on η, and (c_1, c_2) in the interior of $\mathcal{C}(T)$, so that

$$\frac{C\rho}{\eta} + c_1 - \gamma_3 + \varrho \leq \gamma_1, \qquad \frac{C\rho}{\eta} + c_2 + \varrho \leq \gamma_2.$$

Therefore, since $\gamma_3 \geq 0$ (which is a consequence of the assumptions that $r > p$ and $r \geq q$),

$$|\Omega^n| \gtrsim \alpha_{n,1}^{\gamma_1} \alpha_{n,2}^{\gamma_2} |\Pi(F_n)|^{\gamma_3},$$

and hence

$$\langle T\chi_{E_n}, \chi_{F_n} \rangle \lesssim |E_n|^{1/p} \|\chi_{F_n}\|_{q', r'}.$$

Using this and then Hölder's inequality in (20) (here we use the assumption $q \geq p$), we have

$$\begin{aligned}
\langle T\chi_E, \chi_F \rangle &\lesssim \beta^{-\eta} \sum_n \langle T\chi_{E_n}, \chi_{F_n} \rangle \\
&\lesssim \beta^{-\eta} \sum_n |E_n|^{1/p} \|\chi_{F_n}\|_{q', r'} \\
&\lesssim \beta^{-\eta} \left[\sum_n |E_n|\right]^{1/p} \left[\sum_n \|\chi_{F_n}\|_{q', r'}^{q'}\right]^{1/q'} \\
&\leq \beta^{-\eta} |E|^{1/p} \|\chi_F\|_{q', r'}.
\end{aligned}$$

In the last inequality we have used the bounded overlap property of the collections $\{E_n\}$ and $\{F_n\}$. Since $\eta > 0$ can be chosen to be arbitrarily small, this finishes the proof of Theorem 1.

4. Proof of Lemma 1

We refer the reader to [2] for the proof of (i) and (ii).

Let B be the ball $B(z_0, \delta_1, \delta_2)$. Note that $\{e^{sV_1} z_0 : |s| < \delta_1\} \subset B$ and by (1), $|\Pi\pi_2(\{e^{sV_1} z_0 : |s| < \delta_1\})| \approx \delta_1$. Now, we prove that $|\Pi\pi_2(B)| \lesssim \delta_1$. Let $z \in B$ and let φ be a curve connecting z_0 to z as in Definition 1. Let

$\psi = \Pi \circ \pi_2 \circ \varphi$. Since V_2 is contained in the kernel of $D\pi_2$, $|a_1(t)| < \delta_1$ and $|V_1| \lesssim 1$, we have

$$|\psi'(t)| \lesssim \delta_1.$$

Therefore $|\Pi\pi_2(B)| \lesssim \delta_1$. This proves (iii).

We have actually shown that the 1-dimensional measure of $B(z_0, 2\delta_1, 2\delta_2) \cap \pi_1^{-1}(x)$ is $\approx \delta_1$ for every $x \in \pi_1(B)$. This also proves (ii) for $j = 1$. In fact it implies that if A is a subset of a k-dimensional smooth submanifold of X, then δ_1 times the k-dimensional measure of A is comparable to the $k+1$-dimensional measure of $\pi_1^{-1}(A) \cap B$. The corresponding statement holds for $j = 2$, as well.

To prove (iv), define $f(t, \delta_1, \delta_2) = |\pi_2^{-1}\Pi^{-1}(t) \cap B(z_0, \delta_1, \delta_2)|$, where $|\cdot|$ signifies the d-dimensional measure of a subset of Z. To simplify notation we suppose that $\Pi(\pi_2(z_0)) = 0$; this can be achieved by a change of coordinates. Note that if $A \subset B(z_0, \delta_1, \delta_2)$ then for $s < \delta_1$, $e^{sV_1}A \subset B(z_0, 2\delta_1, 2\delta_2)$. Choose $t_0 \in [-\delta_1, \delta_1]$ so that $f(t_0, \delta_1, \delta_2) = \max_{|t| \le \delta_1} f(t, \delta_1, \delta_2)$. Then

$$|B(z_0, 2\delta_1, 2\delta_2)| \gtrsim \int_{|s| \le \delta_1} |e^{sV_1} f(t_0)| ds \gtrsim \delta_1 f(t_0).$$

Now $|B(z_0, \delta_1, \delta_2)| \gtrsim |B(z_0, 2\delta_1, 2\delta_2)|$; the proof given by Nagel, Stein, and Wainger [11] of the volume doubling property carries over to two-parameter balls with (θ, A)–weakly comparable radii, with a bound depending on θ and on A but not on δ_1, δ_2, z_0. We conclude that $f(t) \lesssim |B(z_0, \delta_1, \delta_2)|/\delta_1$ whenever $|t| \le \delta_1$.

Similar reasoning shows that if $|t|, |t'| \le \delta_1$ then $f(t', \delta_1, \delta_2) \le Cf(t, 3\delta_1, 3\delta_2)$. Since $\Pi(\pi_2(B(z_0, \delta_1, \delta_2)) \subset [-\delta_1, \delta_1]$, there exists some $t' \in [-\delta_1, \delta_1]$ for which $f(t', \delta_1, \delta_2) \gtrsim |B(z_0, \delta_1, \delta_2)|/\delta_1$. By combining these two observations, we conclude the reverse inequality $f(t, 3\delta_1, 3\delta_2) \gtrsim |B(z_0, \delta_1, \delta_2)|/\delta_1$ whenever $|t| \le \delta_1$.

The lower bound for $\|\chi_{\pi_2(B)}\|_{q',r'}$ in (iv) follows from this together with the observation that $f(t) \approx |\pi_2(f(t))|\delta_2$. The upper bound follows in the same way from the upper bound $f(t) \lesssim |B(z_0, \delta_1, \delta_2)|/\delta_1$.

Finally, conclusion (v) follows from (ii) and (iv). □

5. On the hypothesis $p \le q \le r$

One reason why the restriction $r \ge q \ge p$ in Theorem 1 is natural is as follows.

Proposition 3. *Suppose that $|B(z, \delta_1, \delta_2)|$ is comparable to $|B(z', \delta_1, \delta_2)|$, uniformly for all z, z' and all weakly comparable δ_1, δ_2. Then all valid mixed norm inequalities for T are implied by the conjectured inequalities. That is:*
(i) If T is of restricted weak mixed type (p, q, r) with $r > p > q$, then $(p, p, r) \in P(T)$.
(ii) If T is of restricted weak mixed type (p, q, r) with $q > r > p$, then (p, q, r) is an interpolant between $(1, \infty, 1)$ and some $(p_1, q_1, r_1) \in P(T)$.

In case (i), the restricted weak mixed type bound for (p,p,r) implies that for (p,q,r) by Hölder's inequality, since we are working in a bounded region and $q<p$. In case (ii), the conclusion is that (p^{-1},q^{-1},r^{-1}) belongs to the line segment with endpoints $(1,0,1)$ and $(p_1^{-1},q_1^{-1},r_1^{-1})$. Since any generalized Radon transform T is of strong type $(1,\infty,1)$, the restricted weak mixed type (p,q,r) inequality follows from the (p_1,q_1,r_1) inequality by interpolation.

Proof. (i) If $(p,p,r)\notin P(T)$ then there exist θ,A and a sequence of balls $B_n(z_n,\delta_{n,1},\delta_{n,2})$ with $\delta_{n,1}\sim_{(\theta,A)}\delta_{n,2}$ satisfying

$$|B_n|^{\frac{1}{r}-\frac{1}{p}}\delta_{n,1}^{\frac{2}{p}-\frac{1}{r}}\delta_{n,2}^{1-\frac{1}{r}}\to\infty,$$

as $n\to\infty$. Choose $N_n\approx\delta_{n,1}^{-1}$ balls of size comparable to B_n with disjoint projections under $\Pi\circ\pi_2$; this is possible by Lemma 1. Let U_n be the union of these balls. Note that $|\pi_1(U_n)|\lesssim N_n|\pi_1(B_n)|$.

$$\begin{aligned}\frac{|U_n|}{|\pi_1(U_n)|^{1/p}\|\chi_{\pi_2(U_n)}\|_{q',r'}}&\gtrsim\frac{N_n|B_n|}{\left[N_n\frac{|B_n|}{\delta_{n,1}}\right]^{1/p}\left[\frac{|B_n|}{\delta_{n,1}\delta_{n,2}}\right]^{1/r'}\delta_{n,1}^{1/q'}N_n^{1/q'}}\\&\approx\left(|B_n|^{\frac{1}{r}-\frac{1}{p}}\delta_{n,1}^{\frac{2}{p}-\frac{1}{r}}\delta_{n,2}^{1-\frac{1}{r}}\right)(N_n\delta_{n,1})^{1/q-1/p}\\&\approx|B_n|^{\frac{1}{r}-\frac{1}{p}}\delta_{n,1}^{\frac{2}{p}-\frac{1}{r}}\delta_{n,2}^{1-\frac{1}{r}}\to\infty,\end{aligned}$$

as $n\to\infty$. Thus, T can not be of restricted weak mixed type (p,q,r).

(ii) For $s\in[0,1]$ define p_1,q_1,r_1 by

$$\frac{1}{p}=1-s+\frac{s}{p_1},\quad\frac{1}{q}=\frac{s}{q_1},\quad\frac{1}{r}=1-s+\frac{s}{r_1}.$$

A bit of algebra shows that $r_1\ge q_1\ge p_1$ if and only if $\frac{1}{q}+\frac{1}{p'}\le s\le\frac{1}{q}+\frac{1}{r'}$, and moreover that $0<\frac{1}{q}+\frac{1}{p'}\le\frac{1}{q}+\frac{1}{r'}<1$ under our assumption that $q>r>p$. Thus it is possible to choose $s\in(0,1)$ so that $r_1\ge q_1\ge p_1$. Fix such a parameter s.

If $(p_1,q_1,r_1)\in P(T)$ then we have the conclusion of case (ii). Otherwise there exist θ,A and a sequence of balls $B_n=B_n(z_n,\delta_{n,1},\delta_{n,2})$ with $\delta_{n,1}\sim_{(\theta,A)}\delta_{n,2}$ such that

$$|B_n|^{\frac{1}{r_1}-\frac{1}{p_1}}\delta_{n,1}^{\frac{1}{p_1}+\frac{1}{q_1}-\frac{1}{r_1}}\delta_{n,2}^{1-\frac{1}{r_1}}\to\infty.$$

Note that for these balls

$$\begin{aligned}\frac{|B_n|}{|\pi_1(B_n)|^{1/p}\|\chi_{\pi_2(B_n)}\|_{q',r'}}&\approx|B_n|^{\frac{1}{r}-\frac{1}{p}}\delta_{n,1}^{\frac{1}{p}+\frac{1}{q}-\frac{1}{r}}\delta_{n,2}^{1-\frac{1}{r}}\\&=\left(|B_n|^{\frac{1}{r_1}-\frac{1}{p_1}}\delta_{n,1}^{\frac{1}{p_1}+\frac{1}{q_1}-\frac{1}{r_1}}\delta_{n,2}^{1-\frac{1}{r_1}}\right)^s\to\infty.\end{aligned}$$

Thus T can not be of restricted weak mixed type (p,q,r). □

Case (ii) is valid for all operators T, without the hypothesis that balls of equal bi-radii have uniformly comparable measures.

References

[1] M. Christ, *Convolution, curvature, and combinatorics. A case study*, Internat. Math. Research Notices 19 (1998), 1033-1048.

[2] ______, *Lebesgue space bounds for one-dimensional generalized Radon transforms*, preprint.

[3] ______, *Quasi-extremals for a Radon-like transform*, preprint.

[4] M. Christ and M. B. Erdoğan, *Mixed norm estimates for a restricted X-ray transform*, J. d'Analyse 87 (2002), 187-198.

[5] M. Christ, A. Nagel, E. M. Stein, S. Wainger, *Singular Radon transforms: Analysis and geometry*, Ann. of Math. (2) 150 (1999), 489-577.

[6] M. B. Erdoğan, *Mixed norm estimates for a restricted X-ray transform in* $\mathbf{R}^4$ *and* $\mathbf{R}^5$, Internat. Math. Research Notices 11 (2001), 575-600.

[7] A. Greenleaf and A. Seeger, *Fourier integral operators with fold singularities*, J. Reine Angew. Math. 455 (1994), 35-56.

[8] ______, *Fourier integral operators with cusp singularities*, Amer. Jour. Math. 120 (1998), 1077–1119.

[9] ______, *On X-ray transforms for rigid line complexes and integrals over curves in* $\mathbf{R}^4$, Proc. Amer. Math. Soc. 127 (1999), 3533-3545.

[10] A. Greenleaf, A. Seeger, S. Wainger, *Estimates for generalized Radon transforms in three and four dimensions*, Analysis, geometry, number theory: the mathematics of Leon Ehrenpreis (Philadelphia, PA, 1998), 243-254, Contemp. Math., 251, Amer. Math. Soc., Providence, RI, 2000.

[11] A. Nagel, E. Stein, S. Wainger, *Balls and metrics defined by vector fields I. Basic properties,* Acta. Math. 155 (1985), 103-147.

[12] A. Seeger, *Radon transforms and finite type conditions*, J. Amer. Math. Soc. 11 (1998), 869-897.

[13] T. Tao and J. Wright, L^p *improving bounds for averages along curves*, J. Amer. Math. Soc. 16 (2003), 605-638.

[14] T. Wolff, *A sharp bilinear cone restriction estimate*, Ann. of Math. (2) 153 (2001), 661-698.

Department of Mathematics, University of California, Berkeley, CA 94720-3840

Department of Mathematics, University of Illinois, Urbana, IL 61801