NONUNIQUENESS OF WEAK SOLUTIONS OF THE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. Generalized solutions of the Cauchy problem for the one-dimensional periodic nonlinear Schrödinger equation, with cubic or quadratic nonlinearities, are not unique. For any $s < 0$ there exist nonzero generalized solutions varying continuously in the Sobolev space H^s, with identically vanishing initial data.

1. Introduction

The Cauchy problem for the one-dimensional periodic cubic nonlinear Schrödinger equation is

$$\text{(NLS)} \qquad \begin{cases} iu_t + u_{xx} + \omega |u|^2 u = 0 \\ u(0,x) = u_0(x) \end{cases}$$

where $x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $t \in \mathbb{R}$, and the parameter ω equals ± 1. Bourgain [2] has shown this problem to be wellposed in the Sobolev space H^s for all $s \geq 0$. That is, there exists a Banach space $Y \subset C^0([0,T], H^s(\mathbb{T})) \cap L^3([0,T] \times \mathbb{T})$ such that for any $u_0 \in H^s$ there exists a solution $u \in Y$, and solutions within the class Y are unique. Here T depends on the H^s norm of the initial datum. An alternative proof of existence of solutions in $C^0([0,T], H^s(\mathbb{T}))$ for $s \geq 0$, without any uniqueness assertion, was recently given [5].

On the other hand, the wellposedness theory breaks down in Sobolev spaces of negative order. For $s < 0$ the mapping from smooth data to solutions fails to be uniformly continuous [4] with respect to the H^s norm, and is unstable in stronger senses [8] as well. For $s < -\frac{1}{2}$, for any $\varepsilon > 0$ there exists a solution[1] $u \in C^\infty$ satisfying $\|u\|_{C^0([0,\varepsilon], H^s)} > \varepsilon^{-1}$ with initial datum satisfying $\|u(0,\cdot)\|_{H^s} \leq \varepsilon$.

There remains the question of unconditional uniqueness, that is, uniqueness of solutions belonging to $C^0([0,T], H^s)$, without further restrictions. As it stands, this question is not well formulated, because of the lack of any well-defined product for general sufficiently singular distributions. In particular, the information $u \in C^0([0,T], H^s)$ alone is insufficient to ensure that the nonlinear expression $|u|^2 u$ has a natural interpretation as a space-time distribution. When s is sufficiently large this expression makes sense, and solutions in $C^0([0,T], H^s)$ are then well known to be unique. More refined work has established sufficient conditions on s for unconditional uniqueness for various equations; see for instance [11] and references cited there.

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[1] The construction of these solutions in [8] does *not* permit any passage to the limit to obtain nonvanishing solutions with vanishing initial data.

In this note we establish nonuniqueness of solutions to the Cauchy problem for the (periodic, cubic) nonlinear Schrödinger equation and its variants with quadratic nonlinearities in classes $C^0([0,T],H^s(\mathbb{T}))$ for $s<0$. While the paper focuses on one prototypical equation and some of its variants, the underlying construction is quite general. Two caveats must be admitted: (i) The solutions constructed are sufficiently singular that the meaning of the nonlinear terms in the equations must be clarified before it can be discussed whether the differential equation is actually satisfied. We prove that the required nonlinear expressions have reasonable and canonical interpretations, and that the differential equations hold under these interpretations. (ii) In the cubic case, the differential equation is modified slightly. The resulting modified Cauchy problem (NLS*) has a reasonable existence theory with uniformly continuous dependence on initial data, in a natural but weak sense, for a *wider* class of function spaces than does (NLS). See below for more precise discussions of these two points.

For (NLS*) there exist certain function spaces $\mathcal{H}$ such that rather canonical solutions in $C^0([0,T],\mathcal{H})$ exist for all initial data $u_0\in\mathcal{H}$, with uniformly continuous dependence upon initial data, yet solutions in $C^0([0,T],\mathcal{H})$ fail to be unique. The same holds for the nonlinear Schrödinger equation with certain quadratic nonlinearities, in Sobolev spaces H^s for all strictly negative s.

2. Results

2.1. Definitions.

Our modified Cauchy problem is

$$\text{(NLS}^*\text{)}\qquad \begin{cases} iu_t+u_{xx}+\omega\mathbf{N}(u)=0\\ u(0,x)=u_0(x)\end{cases}$$

where

$$\mathbf{N}(u)=\big(|u|^2-2\mu(|u|^2)\big)u \tag{2.1}$$

$$\mu(f)=(2\pi)^{-1}\int_{\mathbb{T}}f(x)\,dx. \tag{2.2}$$

$\mu(|u(t,\cdot)|^2)$ is independent of t for all sufficiently smooth solutions; modifying the equation in this way merely introduces a unimodular scalar factor $e^{2i\mu t}$, where $\mu=\mu(|u_0|^2)$. It is always assumed that $\omega\neq 0$, so that the equation is genuinely nonlinear. For parameters $s<0$, $\mu(|u_0|^2)$ is not defined for typical $u_0\in H^s$, but of course the same goes for $|u_0(x)|^2$. Subtracting $2\mu(|u|^2)u$ makes the equation better behaved, as discussed below; it contributes to the nonuniqueness of solutions by making it possible to reasonably interpret the modified differential equation for a wider class of distributions than the unmodified equation, but does not directly produce any wild behavior.

We will work with the partial Fourier transform, which is defined for smooth functions $f(t,x)$ by

$$\widehat{f}(t,n)=(2\pi)^{-1}\int_{\mathbb{T}}f(t,x)e^{-inx}\,dx\quad\text{for } n\in\mathbb{Z}, \tag{2.3}$$

and is extended to distributions by continuity.

Definition 2.1. A sequence of Fourier cutoff operators is any sequence of linear operators $(\mathcal{P}_N)_{N\in\mathbb{N}}$ which act on $\mathcal{D}'(\mathbb{T})$, and are of the Fourier multiplier form $\widehat{\mathcal{P}_N f}(n)=m_N(n)\widehat{f}(n)$ where the functions $m_N:\mathbb{Z}\to\mathbb{C}$ each have finite support, are uniformly bounded, and satisfy $\lim_{N\to\infty}m_N(k)=1$ for all $k\in\mathbb{Z}$.

Let $\mathcal{N}$ be some nonlinear functional acting on functions of (t,x).

Definition 2.2. Let $u \in \mathcal{D}'((0,1)\times\mathbb{T})$ be a distribution. $\mathcal{N}(u)$ is said to exist and to equal $v \in \mathcal{D}'((0,1)\times\mathbb{T})$ if for every sequence $(\mathcal{P}_N)$ of Fourier cutoff operators,

$$\lim_{N\to\infty} \mathcal{N}(\mathcal{P}_N u) = v \tag{2.4}$$

in the topology of $\mathcal{D}'((0,1)\times\mathbb{T})$.

We emphasize that (2.4) is to hold for every sequence $(\mathcal{P}_N)$, not merely for one sequence. Under general theories of multiplication of distributions [1],[9], products of the objects discussed are always defined, but these products depend on the choice of approximating truncation operators. One could require still more of u by replacing Fourier cutoff operators by an appropriate class of pseudodifferential operators implementing cutoffs in phase space rather than merely in frequency space; we have not investigated this more restrictive notion of existence for the solutions constructed in this paper.

Definition 2.3. $u \in C^0([0,1], H^s(\mathbb{T}))$ will be said to be a weak solution of (NLS*) in the extended sense if $u(0,\cdot) = u_0$, $\mathbf{N}(u)$ exists in the sense of Definition 2.2, and u satisfies $iu_t + u_{xx} + \mathbf{N}(u) = 0$ in the distribution sense in $(0,1)\times\mathbb{T}$ with this interpretation of $\mathbf{N}(u)$.

See [10] for some discussion of this and related, less restrictive, notions of weak solutions.

For any function space $\mathcal{H} = \mathcal{H}(\mathbb{T})$, $C^{-1}([0,1],\mathcal{H})$ will denote the space of all space-time distributions $F(s,x)$ such that $\tilde{F}(t,x) = \int_0^t F(s,x)\,ds$ belongs to $C^0([0,1],\mathcal{H})$, and $\|F\|_{C^{-1}([0,1],\mathcal{H})}$ is the "norm" $\max_{t\in[0,1]} \|\tilde{F}(t,\cdot)\|_{\mathcal{H}}$.

The construction will rely on solutions of the inhomogeneous problem

$$\begin{cases} iv_t + v_{xx} + \omega\mathbf{N}(v) = F \\ v(0,x) = v_0(x). \end{cases} \tag{2.5}$$

We refer to F as a driving force. Constructions of Scheffer [15] and Shnirelman [16] of nonunique solutions for the Euler equation have also utilized solutions of inhomogeneous equations. Our construction is somewhat related to that of Shnirelman.

2.2. **Nonuniqueness for the cubic nonlinearity.**

Theorem 2.1. *For any $s < 0$ and $\omega \neq 0$, there exists a space-time distribution $u \in C^0([0,1], H^s)$, not identically vanishing, which is a weak solution of* (NLS*) *in the extended sense, with initial datum $u_0 \equiv 0$. Moreover, the limit* (2.4) *defining $e^{-it\Delta}\mathbf{N}(u)$ exists in the $C^{-1}([0,1], H^s)$ norm.*

It can be shown by an elaboration of the proof that for any initial datum with $\widehat{u_0} \in \ell^1$, there exist $T > 0$ and two distinct weak solutions in $C^0([0,T], H^s)$ of (NLS*). Similar extensions are possible for all theorems stated below.

The solution u qualifies as a solution in a second sense: There exist sequences of functions $f_n \in C^\infty([0,1]\times\mathbb{T})$, such that $e^{-i\Delta t} f_n(t,x) \to 0$ in $C^{-1}([0,1], H^s)$ norm as $n\to\infty$, and solutions $u_n \in C^0([0,1], H^1)$ of (2.5) with driving forces f_n and initial data $u_0 \equiv 0$, such that $u_n \to u$ in $C^0([0,1], H^s)$ norm as $n \to \infty$.

While Theorem 2.1 concerns rather irregular weak solutions, the essence of the construction is the following approximation result for smooth solutions of the inhomogeneous problem.

Proposition 2.2. *Let $s < 0$ and $\omega \neq 0$. Suppose that $u \in C^\infty([0,1] \times \mathbb{T})$, and that each Fourier coefficient $\widehat{u}(t,n)$ vanishes to infinite order as $t \to 0^+$. Then for any $\varepsilon > 0$ there exist $v, F \in C^\infty([0,1] \times \mathbb{T})$, each of whose Fourier coefficients vanishes to infinite order as $t \to 0^+$, such that v is a solution of the inhomogeneous Cauchy problem* (2.5) *with driving force F, with bounds*

$$\|v - u\|_{C^0([0,1],H^s)} \leq \varepsilon \tag{2.6}$$

$$\|e^{-it\Delta}F\|_{C^{-1}([0,1],H^s)} \leq \varepsilon. \tag{2.7}$$

The other theorems stated below are based on analogous facts.

2.3. **Earlier nonuniqueness results.** Theorem 2.1 should be contrasted with the examples of Scheffer [15] and Shnirelman [16] of nonunique weak solutions of the (periodic, two-dimensional) incompressible Euler equation in $C^0([0,T],H^0)$. The notion of a weak solution is less problematic in that framework, for the nonlinear term $v \cdot \nabla v$ is well-defined as a space-time distribution, under the usual straightforward interpretation via integration by parts, for any $v \in C^0([0,T],H^0)$.

A result related to nonuniqueness for the nonlinear Schrödinger equation on the real line has been established by Kenig, Ponce, and Vega [13]: With a Dirac mass as initial datum, either there exists no solution, or there exists more than one solution.[2] Dix [10] has shown nonuniqueness of weak solutions in $C^0(H^s)$ for Burgers' equation, for $s < -\frac{1}{2}$, via the Cole-Hopf transformation, which transforms solutions of the heat equation to solutions of Burgers' equation by taking a logarithm.

2.4. **Nonuniqueness in more restrictive function spaces.** We will also establish, by a slightly more complicated argument, the analogue of Theorem 2.1 for certain less standard function spaces. These are the spaces $\mathcal{F}\ell^p$ for $p \in [1,\infty)$, defined by

Definition 2.4. $\mathcal{F}\ell^p(\mathbb{T}) = \{f \in \mathcal{D}(\mathbb{T}) : \widehat{f}(\cdot) \in \ell^p\}$.

Here $\mathcal{D}(\mathbb{T})$ is the usual space of distributions, and $\mathcal{F}\ell^p$ is equipped with the norm $\|\widehat{f}\|_{\ell^p(\mathbb{Z})}$.

The Cauchy problem (NLS*) in $\mathcal{F}\ell^p$ exhibits certain attributes of wellposedness for all $p \in [1,\infty)$ [5]: For any $R < \infty$ there exists $T > 0$ such that the solution operator $u_0 \mapsto u(t,x)$, defined initially for all $u_0 \in H^1$, is uniformly continuous (even real analytic) as a mapping from $\{u_0 \in H^1 : \|u_0\|_{\mathcal{F}\ell^p} \leq R\}$, equipped with the $\mathcal{F}\ell^p$ topology, to $C^0([0,T],\mathcal{F}\ell^p)$. Moreover the mapping $u_0 \mapsto u$ defined by extending this mapping from the dense subspace to all of $\mathcal{F}\ell^p$ is actually real analytic, and the function $u(t,x)$ thus defined is a weak solution of the differential equation in the extended sense. The unmodified Cauchy problem (NLS) lacks these features for all $p > 2$; the modified equation is better behaved.

Theorem 2.3. *Let $p > 2$ and $\omega \neq 0$. There exists a weak solution $u \in C^0([0,1],\mathcal{F}\ell^p)$ of* (NLS*), *in the extended sense, which does not vanish identically but has initial datum $u_0 \equiv 0$. Moreover, the limit* (2.4) *defining $e^{-it\Delta}\mathbf{N}(u)$ exists in the $C^{-1}([0,1],\mathcal{F}\ell^p)$ norm.*

[2] [13] does not address the issue of defining $|u|^2u$, and the number of solutions could conceivably depend on the definition used. What is actually proved is that for any interpretation of $|u|^2u$ that is appropriately invariant under Galilean symmetries of the equation, there exists either no solution, or more than one solution.

2.5. **Quadratic nonlinearities.** Consider next the Cauchy problem

$$\text{(NLS}_2\text{)} \qquad \begin{cases} iu_t + u_{xx} + \omega Q(u) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

where

$$Q(u) = u^2, \ = \bar{u}^2, \text{ or } = |u|^2 - \mu(|u|^2). \tag{2.8}$$

Theorem 2.4. *Let $s < 0$ and $\omega \neq 0$. For the Cauchy problem* (NLS$_2$) *with any of the nonlinearities*[3] (2.8)*, there exists $u \in C^0([0,1], H^s)$ which is a weak solution in the extended sense, does not vanish identically, and has initial datum $u_0 \equiv 0$. Moreover, $\lim_{N\to\infty} e^{-it\Delta}Q(\mathcal{P}_N u)$ exists in the $C^{-1}([0,1], H^s)$ norm for any sequence of operators $\mathcal{P}_N$ satisfying the conditions of Definition 2.1.*

For $Q = u^2$ or $\bar{u}^2$, this Cauchy problem is wellposed in H^s for all $s > -\frac{1}{2}$ [14], in the usual sense; for any initial datum in H^s there exists a solution belonging to a space more restrictive than $C^0([0,1], H^s)$, and within this smaller space the solution is unique. Thus for $s \in (-\frac{1}{2}, 0)$ we have simultaneously wellposedness in H^s in the usual sense, and nonuniqueness of weak solutions in the extended sense in $C^0([0,1], H^s)$.

2.6. **Discussion.** The construction proceeds as follows. We consider a sequence of exact solutions u_ν of the modified Cauchy problem with initial data zero and with driving forces $f_\nu = \sum_{|k|\geq M_\nu} c_{k,\nu}(t)e^{ikx}$, where $M_\nu \to \infty$ as $\nu \to \infty$. To leading order, f_ν contributes $v_\nu(t) = -i\int_0^t e^{i(t-s)\Delta} f_\nu(s)\,ds$ to the solution u_ν. We choose $f_{\nu+1}$ so that $\mathbf{N}(v_{\nu+1}) \approx f_\nu$, modulo a very small remainder; it is essential to work in function spaces $\mathcal{H}(\mathbb{T})$ in which it is possible to simultaneously make $v_{\nu+1}$ small in $C^0([0,1], \mathcal{H})$, and $\mathbf{N}(v_{\nu+1})$ large in $C^{-1}([0,1], \mathcal{H})$. Thus nonuniqueness arises via an infinite cascade of "energy" from high spatial Fourier modes to lower Fourier modes, that is, from small spatial scales to large scales. Our construction and that of Shnirelman [16] have in common both the use of driving forces tending weakly to zero, and the exploitation of this reverse energy cascade.

The motivation for the construction is that if the evolution is viewed as a coupled system of ordinary differential equations for the spatial Fourier coefficients of $\widehat{u}(t,n)$, then because this system has infinite dimension, uniqueness should be expected to fail without some growth restriction as $|n| \to \infty$. The main issues in the construction are then that exponential growth with respect to n must be avoided, and that the inverse energy cascade inevitably produces many undesired terms along with terms useful in the construction, and it is required to make all undesired terms small in order to keep the H^s norm finite, while useful terms are large and prescribed.

2.7. **Extensions, and potential extensions.** Various related results follow in a straightforward way from the same method.

- Let L be any linear operator of the form $\widehat{Lu}(n) = \sigma(n)\widehat{u}(n)$ where σ is real-valued. Then Theorem 2.1 and its proof carry through, nearly verbatim, when the linear term u_{xx} in the differential equation is replaced by Lu. More generally, if σ has nonnegative imaginary part, the construction goes through if rewritten without the substitution (3.3).
- Generalization to higher dimensions is likewise straightforward.

[3]The same conclusion holds for cubic nonlinearities u^3 and $\bar{u}^3$; no modification of the nonlinearity like that in (NLS*) is required.

- Many other nonlinearities can be treated by the same argument.
- In particular, the periodic Korteweg-de Vries equation admits nonunique solutions, in the extended weak sense, in $C^0(H^s)$ for all $s < 0$. This contrasts with the work of Kappeler and Topalov [12], who have proved existence of quite canonical solutions in $C^0(H^s)$, which depend continuously on initial data in H^s for all $s > -1$. These "solutions" were only proved to satisfy the PDE in the quite weak sense of being limits in $C^0(H^s)$ of C^∞ solutions. Our construction shows that if this notion of solution is liberalized by allowing limits of smooth solutions of inhomogeneous Cauchy problems with smooth driving forces tending to zero in the natural space $e^{it\Delta}(C^{-1}(H^s))$, then solutions are no longer unique.
- The construction applies to semilinear hyperbolic equations $u_{tt} - \Delta_x u + \mathcal{N}(u) = 0$, for many nonlinearities $\mathcal{N}$.
- The construction appears likely to extend to positive Sobolev exponents s for some equations, for instance $iu_t + u_{xx} + u_x\bar{u} = 0$. However, this has not been verified in detail.

An outstanding nonuniqueness result is the construction, originally by Scheffer and then by Shnirelman, of nonzero weak solutions in $L^2_{x,t}$ for the two-dimensional incompressible Euler equation with initial datum zero. The construction given here applies also to the Euler equation, but merely in $C^0(H^s)$ for s strictly negative.

In a sequel we will prove nonuniqueness for the Cauchy problem for the (incompressible periodic) Navier-Stokes equation in dimensions ≥ 2, for solutions in the extended weak sense in $C^0(H^s)$ for s strictly negative. This does not address the question of uniqueness of weak solutions in $C^0(H^0)$.

One feature of the construction is that it is relatively insensitive to the degree of the (semilinear) nonlinear term, in contrast to the behavior of threshold exponents in well-posedness theorems.

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3. Reformulation as an ordinary differential equation

We reformulate the Cauchy problem (NLS*) as an infinite coupled system of ordinary differential equations for the Fourier coefficients of u. Define

$$\sigma(j,k,l,n) = n^2 - j^2 + k^2 - l^2. \tag{3.1}$$

Written in terms of Fourier coefficients $\widehat{u}_n(t) = \widehat{u}(t,n)$ and $\widehat{F}_n(t) = \widehat{F}(t,n)$, the differential equation $iu_t + u_{xx} + \omega \mathbf{N} u = F$ becomes

$$i\frac{d\widehat{u}_n}{dt} - n^2\widehat{u}_n + \omega \sum_{j-k+l=n} \widehat{u}_j\overline{\widehat{u}_k}\widehat{u}_l - 2\omega\widehat{u}_n\sum_m |\widehat{u}_m|^2 = \widehat{F_n}(t). \tag{3.2}$$

Here the first summation is taken over all $(j,k,l) \in \mathbb{Z}^3$ satisfying the indicated identity, and the second over all $m \in \mathbb{Z}$. The term $-2\omega\widehat{u}_n\sum_m |\widehat{u}_m|^2$ cancels out certain terms of the first sum. Eliminating these and substituting[4]

$$y_n(t) = e^{in^2t}\widehat{u}(t,n), \tag{3.3}$$

[4]This substitution is natural but does not materially simplify the analysis here. For dissipative equations it should of course be avoided.

(3.2) becomes

$$\frac{dy_n}{dt} = i\omega \sum_{j-k+l=n}^{*} y_j \bar{y}_k y_l e^{i\sigma(j,k,l,n)t} - i\omega |y_n|^2 y_n - ie^{in^2 t}\widehat{F_n}(t) \tag{3.4}$$

where the notation $\sum^*_{j-k+l=n}$ means that the sum is taken over all $(j,k,l) \in \mathbb{Z}^3$ for which neither $j = n$ nor $l = n$.

For a sequence a define

$$\|a\|^2_{\ell^2_s} = \sum_{n\in\mathbb{Z}} |a_n|^2 (1+n^2)^s. \tag{3.5}$$

Clearly $y \in C^0([0,T], \ell^2_s)$ if and only if $u \in C^0([0,T], H^s)$, with identical norms.

For any complex-valued sequence z define $\mathcal{N}_{\text{diag}}(t)(z)$ and $\mathcal{N}_{\text{main}}(t)(z)$ to be the sequences whose n-th terms are

$$[\mathcal{N}_{\text{main}}(t)(z)]_n = i\omega \sum_{j-k+l=n}^{*} z_j \bar{z}_k z_l e^{i\sigma(j,k,l,n)t}$$
$$[\mathcal{N}_{\text{diag}}(t)(z)]_n = -i\omega |z_n|^2 z_n.$$

and define

$$\mathcal{N}(z) = \mathcal{N}_{\text{main}}(z) + \mathcal{N}_{\text{diag}}(z).$$

For each $t \in \mathbb{R}$, $\mathcal{N}(t)$ is a nonlinear operator which acts on a numerical sequence $z = (z_n)_{n\in\mathbb{Z}}$, and produces another numerical sequence.

We will work with sequence-valued functions y of t, and $\mathcal{N}(y)$ will denote the sequence-valued function $\mathcal{N}(t)(z)$ where $z = y(t)$. With this notation, (3.4) becomes

$$\frac{dy}{dt} = \mathcal{N}(y) + f \tag{3.6}$$

where

$$f_n(t) = -ie^{in^2 t}\widehat{F_n}(t).$$

We say that a sequence-valued function $h(t) = (h_n(t))_{n\in\mathbb{Z}}$ of $t \in [0,1]$ has support contained in $S \subset \mathbb{Z}$ if $h_n(t) \equiv 0$ for all $t \in [0,1]$, for every $n \notin S$. Thus we may speak of sequence-valued functions with finite supports.

4. The main step

Expressed in terms of Fourier coefficients, Proposition 2.2 becomes

Proposition 4.1. *Let $s < 0$. Let $x \in C^\infty([0,1])$ be a finitely supported sequence-valued function such that $x(t,\cdot)$ vanishes to infinite order as $t \to 0^+$. Then for any $\varepsilon > 0$ there exist finitely supported sequence-valued functions $y, g \in C^\infty([0,1])$ satisfying*

$$\begin{cases} \dfrac{dy}{dt} = \mathcal{N}(y) + g(t) \\ y(t,\cdot) \text{ vanishes to infinite order as } t \to 0^+ \end{cases} \tag{4.1}$$

with

$$\|y - x\|_{C^0([0,1],\ell^2_s)} \le \varepsilon \tag{4.2}$$

$$\|g\|_{C^{-1}([0,1],\ell^2_s)} \le \varepsilon. \tag{4.3}$$

Moreover, for any $M < \infty$, y may be constructed so that $y - x$ and g are supported in $[M, \infty)$.

4.1. **Construction of y.** Define

$$f = \frac{dx}{dt} - \mathcal{N}(x). \tag{4.4}$$

Since x has finite support, so does f. Let S be a finite set in which f is supported, and write $S = \{n_j : 1 \le j \le A\}$ where the n_j are distinct. Choose a finite set $S^\dagger \subset \mathbb{Z} \cap [M, \infty)$, as follows. First choose $m_1 \ge M$, and define m_1' by the equation $2m_1 - m_1' = n_1$. Make m_1 sufficiently large to ensure that $m_1' \ge M$ as well. Choose $m_2 \ge M$ very large relative to m_1, m_1', and define m_2' by $2m_2 - m_2' = n_2$. Then choose $m_3, m_3', m_4, m_4' \ldots$ in that order, satisfying

$$2m_j - m_j' = n_j \text{ for all } 1 \le j \le A, \tag{4.5}$$

and let $S^\dagger = \{m_1, m_1', \cdots, m_A, m_A'\}$. The elements of $S^\dagger$ are to be chosen to satisfy additional constraints:

(1) If $k, l, m \in S^\dagger$ and if $l \notin \{k, m\}$ then $|k - l + m| \ge M$ unless $(k, l, m) = (m_j, m_j', m_j)$ for some j.
(2) If $k, l \in S^\dagger$ and n belongs to the support of x then $|k - n + l| \ge M$. Moreover $|k - l + n| \ge M$ provided that $k \ne l$.
(3) If $k \in S^\dagger$ and m, n belong to the support of x then $|k - m + n| \ge M$ and $|m + k - n| \ge M$.

Since each m_j' is approximately twice as large as m_j, and since the support of x is finite, all these conditions will hold, provided that m_1 is sufficiently large and each subsequent m_j is chosen sufficiently large relative to $m_1, \cdots, m_{j-1}$, while m_j' is defined to be $2m_j - n_j$.

Choose C^∞ functions $\{h_m(t) : m \in S^\dagger\}$ that vanish to infinite order as $t \to 0$ and satisfy

$$i\omega \overline{h_{m_j'}}(t) h_{m_j}^2(t) \equiv \tfrac{1}{2} e^{-i\sigma(m_j, m_j', m_j, n_j)t} f_{n_j}(t) \tag{4.6}$$

for each $n_j \in S$. It is essential that these functions be chosen so that $\max_{m \in S^\dagger} \|h_m\|_{C^0([0,1])}$ is bounded above by a finite quantity depending only[5] on S and on f, not on the choice of $S^\dagger$ itself. Define $h = (h_j(t))_{j \in \mathbb{Z}}$ by $h_j(t) = 0$ for all $j \notin S^\dagger$, and h_j as above for all $j \in S^\dagger$. Define

$$y = x + h. \tag{4.7}$$

4.2. **Remainder terms.** Define

$$g = \frac{dy}{dt} - \mathcal{N}(y). \tag{4.8}$$

Since x, h have disjoint supports, $\mathcal{N}_{\text{diag}}(x + h) = \mathcal{N}_{\text{diag}}(x) + \mathcal{N}_{\text{diag}}(h)$. Consequently

$$g = \big(f - \mathcal{N}_{\text{main}}(h)\big) + \frac{dh}{dt} - \mathcal{N}_{\text{diag}}(h) - \big(\mathcal{N}_{\text{main}}(x + h) - \mathcal{N}_{\text{main}}(x) - \mathcal{N}_{\text{main}}(h)\big). \tag{4.9}$$

The bounds on $y - x$ and g in Proposition 4.1 will now be established. As in other constructions of poorly behaved solutions [4],[6],[7],[8], we work in a regime in which nonlinear effects are more powerful than dispersion.

[5]The C^1 norms of the functions h_m will be finite but must depend on $S^\dagger$. This is due to the dispersive nature of the PDE, and prevents us from making dh/dt small in $C^0(\ell^2_s)$. This is an essential part of the obstruction to extending the construction to positive Sobolev exponents.

Lemma 4.2. *Let x be as in the hypotheses of Proposition 4.1, and let h be constructed as above. Then for any $\varepsilon > 0$ there exists $M < \infty$ such that if $S^\dagger$ is chosen as specified, then*

$$\|h\|_{C^0([0,1],\ell^2_s)} \le \varepsilon \tag{4.10}$$

$$\|\frac{dh}{dt}\|_{C^{-1}([0,1],\ell^2_s)} \le \varepsilon \tag{4.11}$$

$$\|\mathcal{N}_{\mathrm{main}}(x) + \mathcal{N}_{\mathrm{main}}(h) - \mathcal{N}_{\mathrm{main}}(x+h)\|_{C^0([0,1],\ell^2_s)} \le \varepsilon \tag{4.12}$$

$$\|\mathcal{N}_{\mathrm{main}}(h) - f\|_{C^0([0,1],\ell^2_s)} \le \varepsilon \tag{4.13}$$

$$\|\mathcal{N}_{\mathrm{diag}}(h)\|_{C^0([0,1],\ell^2_s)} \le \varepsilon \tag{4.14}$$

Proof. $h_k(t)$ vanishes for all $k \notin S^\dagger$, and is bounded uniformly by a finite constant depending on f, independent of the choice of $S^\dagger$. The cardinality of $S^\dagger$ likewise depends only on x. Since s is strictly negative, it follows that $\|h\|_{C^0([0,1],\ell^2_s)} \le CM^s$.

The bound for $\frac{dh}{dt}$ is merely a restatement of the bound for h. $\mathcal{N}_{\mathrm{diag}}(h)$ is also supported in $S^\dagger$, and the same reasoning as for h applies to it.

$\mathcal{N}_{\mathrm{main}}(h) - f$ is supported on $\{n : |n| \ge M\}$, by 1. The term $\mathcal{N}_{\mathrm{main}}(x) + \mathcal{N}_{\mathrm{main}}(h) - \mathcal{N}_{\mathrm{main}}(x+h)$ is supported in the same set, by 2 and 3. Therefore the same reasoning applies to them and yields the same bound CM^s. □

5. A solution with zero initial datum

5.1. **Construction of the solution.** By induction on $n \in \mathbb{N}$, we construct a sequence of finitely supported C^∞ sequence-valued functions $x^{(n)} \in C^1([0,1])$ which vanish to infinite order as $t \to 0$. To begin, choose $x^{(1)}$ to be smooth, to have finite support, to vanish to infinite order as $t \to 0$, and moreover to have 0-th component satisfying

$$\|x_0^{(1)}\|_{C^0([0,1])} \ge 1. \tag{5.1}$$

For the inductive step, construct $x^{(n+1)} = y$ by applying Proposition 4.1 to $x = x^{(n)}$. Define the increments $h^{(n)} = x^{(n+1)} - x^{(n)}$ and the driving forces $f^{(n+1)} = \frac{dx^{(n)}}{dt} - \mathcal{N}(x^{(n)})$. Then by induction $h^{(n)}$ and hence $x^{(n+1)}$ vanish to infinite order as $t \to 0$. Taking ε to be sufficiently small in the conclusion of the proposition at each step, we obtain bounds

$$\|h^{(n)}\|_{C^0([0,1],\ell^2_s)} \le \delta_n \tag{5.2}$$

$$\|f^{(n+1)}\|_{C^{-1}([0,1],\ell^2_s)} \le \delta_n \tag{5.3}$$

$$\delta_n \le 2^{-n-1} \tag{5.4}$$

and moreover each δ_n may be arranged to be as small as may be desired, relative to any quantity depending only on $x^{(n)}$. Moreover $h^{(n)}$ and $f^{(n+1)}$ are naturally expressed as finite sums of various constituent quantities, discussed in the proof of Proposition 4.1 and in Lemma 4.2, which are also $\le \delta_n$.

Define

$$x = \lim_{n\to\infty} x^{(n)} \in C^0([0,1], \ell^2_s); \tag{5.5}$$

the limit exists because the sequence $(x^{(n)})_{n\in\mathbb{N}}$ is constructed so as to be Cauchy in $C^0([0,1], \ell^2_s)$, as stated in (5.2). Together, (5.1) and (5.2) ensure that the component x_0 does not vanish identically as a function of $t \in [0,1]$, so x is a nonzero element of $C^0(\ell^2_s)$. Because $x^{(n)}(0) \equiv 0$, the same holds for x; that is, x satisfies the desired initial condition at time $t = 0$.

5.2. **Existence of $\mathcal{N}(x)$.** In order to show that x satisfies the desired differential equation, we must first show that $\mathcal{N}(x)$ is well-defined.

Lemma 5.1. *Let $(\mathfrak{m}_N)$ be a uniformly bounded sequence of finitely supported functions from $\mathbb{Z}$ to $\mathbb{C}$, and suppose that $\lim_{N\to\infty} \mathfrak{m}_N(n) = 1$ for all $n \in \mathbb{Z}$. Define*[6] *the operators $\mathcal{P}_N y_n = \mathfrak{m}_N(n) y_n$. Then $\lim_{N\to\infty} \mathcal{N}(\mathcal{P}_N x)$ exists in $C^{-1}([0,1], \ell^2_s)$ norm.*

Two facts will be repeatedly used in the proof of Lemma 5.1. Firstly,

$$\|\mathcal{N}(v) - \mathcal{N}(w)\|_{C^0([0,1],\ell^1)} \le C\|v-w\|_{C^0([0,1],\ell^1)} \cdot \left(\|v\|_{C^0([0,1],\ell^1)} + \|w\|_{C^0([0,1],\ell^1)}\right)^2. \tag{5.6}$$

Secondly, the operators $\mathcal{P}_N$ are uniformly bounded on $C^r([0,1],\mathcal{H})$ for $\mathcal{H} = \ell^1$ and $\mathcal{H} = \ell^2_s$, for $r = 0$ and $r = -1$.

For any N, k,

$$\|\mathcal{N}(\mathcal{P}_N x^{(k)}) - \mathcal{N}(\mathcal{P}_N x)\|_{C^0([0,1],\ell^1)} \le CN^3 2^{-k}, \tag{5.7}$$

since $x^{(k)} - x \le 2^{-k}$ in $C^0([0,1],\ell^\infty)$ norm and $\mathcal{P}_N y$ is supported $[-3N, 3N]$ for any y. Thus for any index J

$$\mathcal{N}(\mathcal{P}_N x) = \mathcal{N}(\mathcal{P}_N x^{(J)}) + \sum_{j=J}^{\infty} \left[\mathcal{N}(\mathcal{P}_N x^{(j+1)}) - \mathcal{N}(\mathcal{P}_N x^{(j)})\right] \tag{5.8}$$

with convergence in the $C^0([0,1],\ell^1)$ norm.

For any fixed k, $\mathcal{P}_N x^{(k)} \to x^{(k)}$ in $C^0([0,1],\ell^1)$ norm since the multipliers $\mathfrak{m}_N$ are uniformly bounded and tend pointwise to 1. Therefore

$$\|\mathcal{N}(\mathcal{P}_N x^{(k)}) - \mathcal{N}(x^{(k)})\|_{C^0([0,1],\ell^1)} \to 0 \text{ as } N \to \infty \tag{5.9}$$

by (5.6).

Lemma 5.2. *If the construction of the sequence $(x^{(n)})_{n\in\mathbb{Z}}$ is carried out so that each δ_n is sufficiently small relative to quantities determined at earlier steps of the construction, then there exists $C < \infty$ such that for all k and all N,*

$$\|\mathcal{N}(\mathcal{P}_N x^{(k+1)}) - \mathcal{N}(\mathcal{P}_N x^{(k)})\|_{C^{-1}([0,1],\ell^2_s)} \le C2^{-k}. \tag{5.10}$$

Proof. Rewrite

$$\begin{aligned} &\mathcal{N}(\mathcal{P}_N x^{(k)} + \mathcal{P}_N h^{(k)}) - \mathcal{N}(\mathcal{P}_N x^{(k)}) \qquad (5.11)\\ &= \left[\mathcal{N}(\mathcal{P}_N x^{(k)} + \mathcal{P}_N h^{(k)}) - \mathcal{N}(\mathcal{P}_N x^{(k)}) - \mathcal{N}(\mathcal{P}_N h^{(k)})\right] + \mathcal{N}_{\text{main}}(\mathcal{P}_N h^{(k)}) + \mathcal{N}_{\text{diag}}(\mathcal{P}_N h^{(k)}). \end{aligned}$$

Now $\mathcal{N}_{\text{diag}}(\mathcal{P}_N h^{(k)})$ can be bounded in $C^0(\ell^2_s)$ norm exactly as was done for $\mathcal{N}_{\text{diag}}(h^{(k)})$ in the proof of Lemma 4.2, up to an additional factor of $\|\mathfrak{m}_N\|^3_{\ell^\infty}$. The same applies to $\mathcal{N}(\mathcal{P}_N x^{(k)} + \mathcal{P}_N h^{(k)}) - \mathcal{N}(\mathcal{P}_N x^{(k)}) - \mathcal{N}(\mathcal{P}_N h^{(k)})$ in comparison with $\mathcal{N}(x^{(k)} + h^{(k)}) - \mathcal{N}(x^{(k)}) - \mathcal{N}(h^{(k)})$.

$\mathcal{N}_{\text{main}}(\mathcal{P}_N h^{(k)})$ breaks up into two parts. First there is the contribution of all 3-tuples $(k,l,m) \in S^{\dagger 3}$, with $l \ne k, m$, that are not of the form (m_j, m'_j, m_j). The same analysis given for $\mathcal{N}_{\text{main}}(h^{(k)}) - f^{(k)}$ in the proof of Lemma 4.2 applies to the sum of these terms, up to the factor of $\|\mathfrak{m}_N\|^3_{\ell^\infty}$. Thus the sum of these terms is again as small as desired in $C^0(\ell^2_s)$ norm.

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[6]An abuse of notation; this $\mathcal{P}_N$ is related to what is called $\mathcal{P}_N$ elsewhere in the paper by conjugation with the spatial Fourier transform.

There remains the contribution of all 3-tuples (m_j, m_j', m_j). Any such 3-tuple contributes exactly $\mathfrak{m}_N(m_j)^2\overline{\mathfrak{m}_N(m_j')}$ times $f^{(k)}_{n_j}(t)$. Because $\mathfrak{m}_N(m_j)^2\overline{\mathfrak{m}_N(m_j')}$ is independent of t, we therefore have the same upper bound in $C^{-1}(\ell^2_s)$ as for $f^{(k)}$ itself, up to the factor $\|\mathfrak{m}_N\|^3_{\ell^\infty}$. □

Lemma 5.1 follows directly from the combination of Lemma 5.2 with (5.8) and (5.9). Henceforth $\mathcal{N}(x)$ is well-defined, via Lemma 5.1.

Corollary 5.3.

$$\mathcal{N}(x^{(k)}) \to \mathcal{N}(x) \text{ in } C^{-1}([0,1],\ell^2_s) \text{ norm as } k \to \infty. \tag{5.12}$$

Proof. By Lemma 5.2,

$$\|\mathcal{N}(\mathcal{P}_N x) - \mathcal{N}(\mathcal{P}_N x^{(k)})\|_{C^{-1}([0,1],\ell^2_s)} \le C2^{-k}, \tag{5.13}$$

uniformly in N. By Lemma 5.1 and (5.9),

$$\mathcal{N}(\mathcal{P}_N x) - \mathcal{N}(\mathcal{P}_N x^{(k)}) \to \mathcal{N}(x) - \mathcal{N}(x^{(k)}) \text{ in } C^{-1}([0,1],\ell^2_s) \text{ norm as } N \to \infty. \tag{5.14}$$

Therefore

$$\|\mathcal{N}(x) - \mathcal{N}(x^{(k)})\|_{C^{-1}([0,1],\ell^2_s)} \le C2^{-k}. \tag{5.15}$$

□

5.3. **A solution of the Cauchy problem.** By definition of $f^{(n)}$, $x^{(n)}(t) = \int_0^t \mathcal{N}(x^{(n)}(s))\,ds + \int_0^t f^{(n)}(s)\,ds$. Since $f^{(n)} \to 0$ in $C^{-1}([0,1],\ell^2_s)$ norm, $x^{(n)} \to x$ in $C^0([0,1],\ell^2_s)$, and $\mathcal{N}(x^{(n)}) \to \mathcal{N}(x)$ in $C^{-1}([0,1],\ell^2_s)$, it follows at once that

$$x(t) = \int_0^t \mathcal{N}(x(s))\,ds. \tag{5.16}$$

Define $u \in C^0([0,1],H^s)$ by

$$\widehat{u}(t,n) = e^{-in^2t}x_n(t). \tag{5.17}$$

Since $x^{(n)}(t,x) \in C^0([0,1],\ell^2_s)$ vanishes identically for $t=0$ and tends to x in $C^0([0,1],\ell^2_s)$ norm, u satisfies the initial condition $u(0,\cdot) \equiv 0$. Lemma 5.1 states in equivalent form that $\mathcal{N}(u)$ exists in the sense of Definition 2.2. (5.16) implies that u is a weak solution in the extended sense of the modified nonlinear Schrödinger equation. This completes the proof of Theorem 2.1. □

6. Variants

6.1. **The analogue for $\mathcal{F}\ell^p$.** The proof of Theorem 2.3 is quite similar to that of Theorem 2.1. The only significant change arises in the proof of Proposition 4.1, for one cannot make $\|h\|_{C^0([0,1],\ell^p)}$ arbitrarily small simply by selecting $S^\dagger \subset [M,\infty)$ for M arbitrarily large, as can be done for $\|h\|_{C^0([0,1],\ell^2_s)}$.

The key now is that with a modification of the set $S^\dagger$ of spatial Fourier modes in the support of the new driving force g, making $\mathcal{N}_{\text{main}}(h) \approx f$ requires a lower bound on h in $C^0([0,1],\ell^2)$ but not in $C^0([0,1],\ell^p)$ for $p>2$. Let $S = \{n_j : 1 \le j \le A\}$ be as in the proof of Proposition 4.1. $S^\dagger$ will now be taken to consist of elements m_0 and $m_{j,i}, m'_{j,i}$ for $1 \le j \le A$ and $1 \le i \le K$ where the new parameter K is to be determined. A large integer $m_0 \in \mathbb{N}$ is chosen first, then $m_{1,1} < m_{1,2} < \cdots < m_{1,K} < m_{2,1} < m_{2,2} < \cdots < m_{2,K} < m_{3,1} < \cdots < m_{A,K}$ are chosen in that order, each sufficiently large relative to all its predecessors for later purposes, and then the quantities $m'_{j,i}$ are uniquely determined by the relations

$$m_0 + m_{j,i} - m'_{j,i} = n_j \text{ for all } j, i. \tag{6.1}$$

If m_0 is chosen so that $m_0 - n_j > 0$ for all j then there is no obstruction to choosing $m_{j,i}, m'_{j,i}$ so that this equation holds and $m_0, m_{j,i}, m'_{j,i}$ are three distinct integers.

$h_{m_0}(t)$ is defined to be the constant function $c\varepsilon$ where ε is the small quantity in the conclusion of the Proposition, and c is some sufficiently small fixed constant. Coefficients $\{h_m : m \in S^\dagger\}$ are chosen to be C^1 functions satisfying

$$i\omega h_{m_{j,i}} \overline{h_{m'_{j,i}}} h_{m_0}(t) \equiv \tfrac{1}{2} K^{-1} e^{-i\sigma(m_j, m'_j, m_j, n_j)t} f_{n_j}(t) \tag{6.2}$$

$$\|h_m\|_{C^0([0,1])} \le C\big(\varepsilon^{-1} K^{-1} \|f_{n_j}\|_{C^0}\big)^{1/2} \quad \text{if } m = m_{j,i} \text{ or } m = m'_{j,i}. \tag{6.3}$$

for each $1 \le j \le A$ and each $1 \le i \le K$. If p is strictly greater than 2 then for any given $\delta > 0$, $\{h_{j,i}\}$ can be made to satisfy

$$\Big(\sum_{m \ne m_0 \in S^\dagger} \|h_m\|_{C^0}\Big)^{1/p} \le \delta, \tag{6.4}$$

by choosing K to be sufficiently large as a function of ε, δ, for the factor of $K^{1/p}$ arising from the number of terms on the left-hand side is more than compensated for by the factor of $K^{-1/2}$ in (6.3), and this allows us to absorb the factor $\varepsilon^{-1/2}$ in (6.3). The remainder of the proof of Proposition 4.1 is unchanged. However the statement of the proposition must be modified slightly; the construction makes g small in $C^{-1}([0,1], \ell^\infty)$, but not in $C^{-1}([0,1], \ell^p)$. Repeated applications of the Proposition establish Theorem 2.3, just as for Theorem 2.1. □

6.2. **Quadratic nonlinearities.** Consider the nonlinearity $Q(u) = u^2$; the discussion will apply to $\bar{u}^2$ and $|u|^2 - \mu(|u|^2)$ with very minor changes which are left to the reader. If $S = \{n_j : 1 \le j \le A\}$ then we set $S^\dagger = \{m_j, m'_j : 1 \le j \le A\}$ where $m_j + m'_j = n_j$ and $|m_j|, |m'_j| \ge M$ for all j. The conditions on $\{h_m\}$ now become

$$i\omega h_{m_j}(t) h_{m'_j}(t) \equiv \tfrac{1}{2} e^{-i(m_j^2 + {m'_j}^2 - n_j^2)t} f_{n_j}(t). \tag{6.5}$$

By choosing m_1 sufficiently large and then $|m_j|$ sufficiently large relative to $|m_{j-1}|$ we may ensure that the analogue of Lemma 4.2 holds. The rest of the argument is unchanged. □

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