The $\bar{\partial}$–Neumann Problem, Magnetic Schrödinger Operators, and the Aharonov-Bohm Effect

Michael Christ
University of California, Berkeley

Subject:

• Compactness estimates for the $\bar{\partial}$–Neumann problem in several complex variables.

• Comparison of ground state energies of Pauli/magnetic versus electric Schrödinger operators.

Joint work with $\boxed{\text{Siqi Fu}}$

April 2003

I. The $\bar\partial$–Neumann problem. Let $\Omega \Subset \mathbb{C}^n =$ a (pseudoconvex) open domain with C^∞ boundary.

A basic problem is to solve $\boxed{\bar\partial u = f}$ in Ω, where u is a function, f is a 1-form, and f satisfies the necessary condition $\bar\partial f \equiv 0$. Seek boundary regularity of u.

In general the solution is highly nonunique; general solutions possess no boundary regularity. However there is a **canonical solution**, the one with minimal L^2 norm.

The canonical solution is $\bar\partial^* v$ where v satisfies a second-order **noncoercive** boundary value problem for Laplace's equation.

v has n components and is subjected to one Dirichlet-type boundary condition and $n-1$ **complex** Neumann-type conditions. By the method of boundary reduction (Calderón and others) this boils down to a nonelliptic second-order equation on the boundary.

In $\mathbb{C}^2$ the boundary equation

$$\Box_b u = f$$

is well modeled by equations of the type

$$(X^2 + Y^2)u = f,$$

where X, Y are 2 independent real vector fields on a real 3-dimensional manifold.

X, Y may **or may not** satisfy the *bracket condition* of Hörmander and Kohn. This distinguishes domains of **finite type** from those of **infinite type**. This talk is about **very** degenerate situations.

(Bracket condition: The Lie algebra spanned by $X, Y, [X, Y], [X, [X, Y]]$ and all Lie brackets of all orders spans the tangent space at each point of $\partial\Omega$.)

In higher dimensions the boundary equation is a system and is genuinely more subtle.

There's an extensive literature on *hypoelliptic-ity* – whether the canonical solution is smooth wherever the datum is. "Smooth" can mean C^∞, C^ω, H^s, Hölder, L^p, ...

Global Regularity

An important problem for infinite type C^∞ (pseudoconvex) domains is **global regularity**:

$$f \in C^\infty(\overline{\Omega}) \overset{?}{\Rightarrow} u \in C^\infty(\overline{\Omega})$$

for the canonical solution of $\bar{\partial} u = f$.

When this holds, biholomorphic mappings between domains extend smoothly to the boundaries (Bell-Ligocka), making it possible to (in principle) distinguish biholomorphism classes of domains via geometric boundary invariants.

It does hold for wide classes of domains,
- including all *convex* ones (Boas-Straube)
- but it fails in general (Christ).

Compactness

A very basic property is **compactness**:

$$\{u : \|\bar{\partial}u\|^2_{L^2} + \|\bar{\partial}^* u\|^2_{L^2} \leq 1\}$$

is precompact in $L^2(\Omega)$.

Equivalently, in the boundary interpretation, compactness means

$\{u : \|\Box_b u\|_{L^2(\partial\Omega)} \leq 1\}$ is precompact in $L^2(\partial\Omega)$.

This holds whenever all u are even infinitesimally smoother than $\Box_b u$.
It is a purely <u>local</u> property of the boundary.

Compactness is a *local* property of the boundary, and should be more robust than global regularity.

A sufficient condition for compactness is **property (P)** (Catlin, Sibony): For any $M < \infty$ there exists a C^2 function h

$$\text{defined where distance}(z, \partial\Omega) < \epsilon_M$$

satisfying

$$\sup |h| \leq 1$$

but

$$\frac{\partial^2 h}{\partial z \partial \overline{z}} \geq M.$$

Hence global geometric/topological issues **seem** not to be relevant. But we'll see later in the talk that this is completely wrong . . . ; this problem is intrinsically concerned with analysis on non-manifolds with infinite topology . . .

Questions

• Characterize compactness more geometrically and constructively?
• Are there $\boxed{\text{other effects}}$, not captured by (P), which produce compactness?

An obvious crude $\boxed{\text{obstruction}}$ to compactness (in $\mathbb{C}^2$) is the presence of a

$$\boxed{\text{complex disk embedded in } \partial\Omega}.$$

(that is, a piece of one-dimensional complex submanifold of $\mathbb{C}^2$)

Matheos (1997) constructed a C^∞ pseudoconvex Hartogs domain in $\mathbb{C}^2$ which has no such disks, but for which compactness, and hence (P), fail.

Part II. Schrödinger Operators

Hartogs domains in $\mathbb{C}^2$:

$$(z, w) \in \Omega \Rightarrow (z, e^{i\theta} w) \in \Omega \quad \text{for any } \theta \in \mathbb{R}.$$

Applying Fourier decomposition in θ to the boundary operator $X^2 + Y^2$ produces sequence of operators

$$-(\partial_x + in\phi_y)^2 - (\partial_y - in\phi_x)^2 + n\Delta\phi$$

where Δ is the Laplacian in $z = x + iy \in \mathbb{C}^1$
$n \in \mathbb{Z}^+$ is Fourier variable dual to θ and
$\boxed{\Delta\varphi \geq 0}$ ($\Leftrightarrow$ pseudoconvexity).

Electric potential $= n\Delta\phi$.
Magnetic potential $= (n\phi_y, -n\phi_x)$.

This particular relation between electric and magnetic potentials occurs naturally in quantum physics; the *Pauli operator* describing particles with spin in a purely magnetic field in 2D decouples into two components, of which one is $-(\partial_x + in\phi_y)^2 - (\partial_y - in\phi_x)^2$; the ground state energy of this component is comparable to that of ours modulo a factor of 2.

Matheos' analysis based on making electric field $\Delta\phi > 0$ on dense set whose complement is thick (nonempty fine interior in language of potential theory).

For a domain $V_0 \subset \mathbb{C}^1$, with Dirichlet boundary condition: Assume $\Delta\psi \geq 0$.

Define:

$$\lambda_\psi^{\mathsf{e}} = \inf_{\|u\|_{L^2}=1} \int_{V_0} \left[|\nabla u|^2 + \Delta\psi |u|^2 \right]$$

(Electric only lowest eigenvalue)

$$\mathcal{D}_\psi = (\partial_x + i\psi_y, \, \partial_y - i\psi_x)$$

(Magnetic gradient)

$$\lambda_\psi^{\mathsf{m}} = \inf_{\|u\|_{L^2}=1} \int_{V_0} \left[|\mathcal{D}u|^2 + \Delta\psi |u|^2 \right]$$

(Magnetic lowest eigenvalue)

Note that we're discussing the ground state energies for a sequence of systems, not the sequence of all energies for a single system.

As $n \to +\infty$:

• $\lambda^{\mathsf{e}}_{n\varphi} \to \infty \Leftrightarrow$ (P)
(Fu-Straube; usual integration by parts proof).

• $\lambda^{\mathsf{m}}_{n\varphi} \to \infty \Leftrightarrow$ compactness
(nearly a tautology).

Thus we ask: **Main Question.** Can

$$\begin{cases} \lambda^{\mathsf{e}}_{n\varphi} \text{ stay bounded} \\ \qquad \text{while} \\ \lambda^{\mathsf{m}}_{n\varphi} \to \infty \end{cases}$$

as $n \to +\infty$?

B. Simon (1976) proved the *diamagnetic inequality* for spinless bosons, which in our context specializes to

$$\boxed{\lambda^{\mathrm{e}}_{\psi} \leq \lambda^{\mathrm{m}}_{\psi}} \text{ for all subharmonic } \psi.$$

In particular, (P) $\Rightarrow$ compactness.

• $\lambda^{\mathrm{e}}_{n\varphi}$ uniformly bounded essentially means $\exists$ a function u with L^2 gradient,

$$\text{supported on } \mathbb{E} = \{z : \Delta\varphi(z) = 0\}.$$

Thus $\boxed{\mathbb{E} \text{ must be } \textbf{thick}.}$

• $\lambda^{\mathrm{m}}_{n\varphi} \to +\infty$ requires the magnetic Dirichlet form $\int |\mathcal{D}_{n\varphi}u|^2$ to be very strong relative to ∇ as $n \to +\infty$, for functions u supported where $\Delta\varphi = 0$. Thus $\boxed{\mathbb{E} \text{ must have } \textbf{no interior}.}$

This is related to failure of the formerly conjectured *paramagnetic inequality*, which would imply $\lambda^{\mathsf{m}}_{\psi} \leq \lambda^{\mathsf{e}}_{2\psi}$ for all subharmonic ψ. Avron-Simon (1979) gave a counterexample based on the

$\boxed{\text{Aharonov-Bohm effect:}}$

The magnetic <u>potential</u> can exert a physical effect on a system supported in a region <u>disjoint</u> from the support of the magnetic <u>field</u>.

We need to

- Produce a much stronger effect ($\lambda^{\mathsf{m}}_{n\varphi} \to \infty$),
- Do so for **every** large n,
- Do so without introducing a separate electric potential, distinct from the magnetic potential, as [AS] did.

This is a topological effect; it does not occur on simply connected domains.

Part III. Results.

Proposition. There exists subharmonic $\varphi \in C^\infty$ for which

$$\sup_n \lambda^{\mathrm{e}}_{n\varphi} < \infty \quad \text{yet} \quad \boxed{\limsup_{n\to\infty}} \; \lambda^{\mathrm{m}}_{n\varphi} = \infty.$$

Moreover the sequence of bad values of n for which $\lambda^{\mathrm{m}}_{n\varphi}$ remains bounded can be made extremely <u>sparse</u>.

Nonetheless elimination of this sparse sequence is more delicate.

Theorem 1.
There exists subharmonic φ such that

- $\lambda^{\mathsf{e}}_{n\varphi}$ remains bounded as $n \to +\infty$
- $\lambda^{\mathsf{m}}_{n\varphi} \to \infty$ as $n \to +\infty$

and

$$\begin{cases} \varphi \in C^0 \\ \nabla\varphi \in L^2. \end{cases}$$

On the other hand . . .

Theorem 2.
If $\varphi \in C^\infty$ is subharmonic, and <u>if</u>

$$\sup_n \lambda^{\mathrm{e}}_{n\varphi} < \infty$$

<u>then</u>

$$\liminf_{n\to\infty} \lambda^{\mathrm{m}}_{n\varphi} < \infty.$$

For this conclusion it suffices to have $\Delta\varphi$ Hölder continuous of some order.

Recall that we have examples in which $\lambda^{\mathrm{m}}_{n\varphi} \to \infty$ as $n \to \infty$ through a sequence whose complement is extremely sparse.

Corollary. For C^∞ pseudoconvex Hartogs domains in $\mathbb{C}^2$, compactness in the $\bar\partial$-Neumann problem is equivalent to property (P).

Part IVA. First proof.

The diamagnetic inequality $\lambda_\psi^{\mathsf{e}} \leq \lambda_\psi^{\mathsf{m}}$

For any real-valued ψ,

$$|\mathcal{D}_\psi u| \geq \Big| \nabla |u| \Big|$$

a.e. for $u \in C^1$. (Kato; Simon)

This was actually a step in Kato's proof of another inequality. It was Simon who observed that this leads to the diamagnetic inequality $\lambda_\psi^{\mathsf{m}} \geq \lambda_\psi^{\mathsf{e}}$.

A simple situation

Let A be an annulus centered at $z = 0$, and D be the disk inside the inner radius of A. Suppose that $\Delta\psi \equiv 0$ on A. Then

$$\int_A |\mathcal{D}_\psi u|^2 \geq cw^2 \int_A |u(z)|^2 |z|^{-2}$$

where

$$w = \underline{\underline{\text{distance}\left(\int_D \Delta\psi,\ \pi\mathbb{Z}\right)}}.$$

Up to unitary equivalence, the magnetic gradient $\mathcal{D}_\psi$ in A depends only on w modulo $2\pi\mathbb{Z}$; therefore $\lambda^{\mathrm{m}}_{n\psi}$ is <u>quasiperiodic</u> in $n \in \mathbb{R}$ in this example.

This unitary equivalence is realized by conjugating by $e^{i\tilde{\psi}}$, where $\tilde{\psi}$ is a (multiple-valued) harmonic conjugate of ψ. This requires

$$e^{i\tilde{\psi}} \text{ must be single-valued.}$$

On the credit side, this does show that the magnetic effect can be strong provided that distance$(w, 2\pi\mathbb{Z})$ is relatively large compared to the outer radius of the annulus.

This inequality does much of the work for us, though we must deal with perturbations in which $\Delta\psi$ is merely small in $\mathcal{A}$.

- Magnetic gradient has effective strength

$$\gtrsim \frac{\kappa}{|z|}$$

where $\kappa = \text{distance}(w, 2\pi\mathbb{Z})$.

- Ordinary gradient has effective strength

$$\sim \frac{1}{|z|\sqrt{\log R}}$$

where

$$R = \frac{\text{outer radius}}{\text{inner radius}}.$$

Form of example:

- Start with $\Omega_0 =$ disk.

- Construct $\mathbb{E} = \Omega_0 \setminus \cup_{k=1}^{\infty} \cup_j D_j^k$ where $D_j^k = B(z_j^k, \rho_k)$ are many small disks.

- For each k, the centers z_j^k form an ε_k-net in $\Omega_0 \setminus \cup_{m=1}^{k-1} \cup_j D_j^m$.

- $\varepsilon_k = 2^{-k}$ and $\boxed{\rho_k = e^{-k^2 2^{2k}}}$.
(**Tiny** disks essential to controlling electric ground energy.)

- $\Delta\varphi =$ sum of pieces supported on disks D_j^k.

The exponential relation $\rho_k \lesssim e^{-k^2 \varepsilon_k^{-2}}$ is nearly optimal (the extra factor k^2 in the exponent is not).

An essentially optimal trial function looks like

$$\frac{\log \frac{|z - z_j^k|}{\rho_k}}{\log \frac{\varepsilon_k}{\rho_k}}$$

in each of the approximately 2^{2k} annuli

$$\rho_k \leq |z - z_j^k| \leq c_0 \varepsilon_k.$$

This has L^2 gradient, uniformly in k, but vanishes on each boundary component ∂D_j^k.

• Let $N_k = e^{c2^k}$. **Key:** Arrange that for each $n \in [N_k, N_{k+1})$, there exist lots of "good" disks D_j^k with favorable winding numbers:

$$n \int_{D_j^k} \Delta\varphi \in [\tfrac{\pi}{2}, \tfrac{3\pi}{2}] \ \text{ modulo } 2\pi\mathbb{Z}.$$

• More specifically:
Each $\sqrt{\varepsilon_k}$–ball in $\Omega_0 \setminus \cup_{m<k} \cup_j D_j^k$ contains at least one such good disk D_j^k.

Combine:
• Favorable magnetic inequality in a small "annulus" about each good disk D_j^k
• High "density" of good disks.
• Global control of ∇ handles comparatively thin region outside favorable disks.

Lemma.
Let $\mathcal{A} = \{r < |z| < R\}$.

If $u \in C^0(\overline{\mathcal{A}}) \cap W^1(\mathcal{A})$ then

$$\int_{\mathcal{A}} |u|^2 \leq (R^2 - r^2) \log(R/r) \int_{\mathcal{A}} \big|\nabla |u|\big|^2 + (R^2 - r^2) \int_0^{2\pi} |u(re^{i\theta})|^2 \, d\theta.$$

This helps us to extend a favorable estimate in a very small disk, due to locally strong magnetic effect, to a much larger disk, by invoking Kato's inequality.

Part IVB. Second proof.

φ smooth — very special case

Let the "holes" $\{D_j^k\}$ be as in our construction, and $\varphi = \sum_{k,j} \varphi_j^k$ where $\Delta\varphi_j^k$ is supported in D_j^k.

Recall that the winding number

$$(2\pi)^{-1} n \int_{D_j^k} \Delta\varphi \text{ modulo } \mathbb{Z}$$

is the obstruction to "gauging away" the magnetic field generated by φ_j^k.

We wish to show that if $\Delta\varphi$ is Hölder continuous, then the compactness estimate fails;

$$\liminf_{n\to\infty} \lambda_{n\varphi}^{\mathrm{m}} \text{ is finite.}$$

Main point: For any such "smooth" φ, there exists a sequence

$$n_k = O(e^{ck^2 2^{2k}})$$

such that

$$n_k \int_{D_j^l} \Delta\varphi = O(e^{-cl^2 2^{2l}})$$

whenever $l > k$ and

$$\boxed{\text{distance}\left(n_k \int_{D_j^k} \Delta\varphi,\ 2\pi\mathbb{Z}\right) = O(e^{-c'k^2})}$$

for all $l \leq k$ and all j.

This last condition allows us to "gauge away" most of the very large potential $n_k \Delta\varphi$, leaving a small potential which hence creates bounded ground energy.

Proof: Pigeonhole argument. The numbers $n \int_{D_j^k} \Delta\varphi$ modulo $2\pi\mathbb{Z}$ form an arithmetic progression in a torus of dimension $\sim C^{2k}$ ...

Difficulties for general smooth φ

• One is not given a clean decomposition with "holes" D_j^k.

• Pigeonhole argument requires number of holes of radius ρ to be

$$\ll \frac{\log \frac{1}{\rho}}{\log \log \frac{1}{\rho}},$$

but the best that is true in general is

$$O(\rho^{-2}).$$

The problem is that half of Ω_0 could be filled by an enormous number of holes. This would merely force the ground state eigenfunction to live in the other half.

Solution

• Fix nonnegative $u \in L^\infty$ supported where $\Delta\varphi = 0$ with $\nabla u \in L^2$. Use this function as an oracle.

• Holes are defined to be connected components of sublevel sets $\{z : u(z) \leq c\}$.

• This implicitly groups clusters of "topological holes" into single entities, and effectively classifies holes into <u>relevant</u> and irrelevant ones.

• A <u>dangerous</u> hole for Fourier coefficient n is one where $\max \Delta\varphi \gtrsim 1/n$.

Key estimate

- Number of dangerous holes is $O(\log n)$.

Proof: Each hole contains a disk of radius $\rho \gtrsim n^{-C}$. $u \equiv 0$ in that disk, but $u \equiv c$ in boundary of hole. This forces

$$\int_{\text{hole}} |\nabla u|^2 \gtrsim \frac{1}{\log(\rho^{-1})} \ .$$

Summing over all holes gives bound since

$$\|\nabla u\|^2_{\text{total}} = O(1).$$

• For a subsequence of values of $n \to \infty$, the number of dangerous holes is

$$\leq \frac{\log n}{\log\log n \cdot \log\log\log n}.$$

• Pigeonhole to produce favorable winding numbers on all relevant holes.

• To get upper bound on $\lambda^{\mathrm{m}}_{n\varphi}$, use trial function

$$f(x) = \begin{cases} u(x) - c & \text{where } u \geq c. \\ 0 & \text{where } u \leq c. \end{cases}$$