Radon-Like Transforms, Quasi-extremals, and (perhaps) Subalgebraic Sets

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Setup:

• $X, X^\star$ are small open sets in $\mathbb{R}^d, \mathbb{R}^{d^\star}$.

• $Tf(x) = \int_{M_x} f\, d\sigma_x$ where M_x is a submanifold of $X^\star$, and σ_x is a measure on M_x. Both vary C^∞ as x varies. $\boxed{\text{Not a foliation.}}$

• Also assume that formal adjoint of T is an operator of same general type (a transversality condition).

Problem: Relate mapping properties of T, in standard function spaces, to geometry of $\{M_x\}$.

Standard function spaces: Sobolev, L^p, and mixed-norm spaces $L^q(L^r)$. This talk is about L^p only.

This is a quite broad class of operators. Only special cases have been analyzed; and in this talk I'll argue that even the most basic cases haven't been fully analyzed.

We all know examples:

- Radon transform.
- x-ray transform.
- Convolution with surface measure on a submanifold of $\mathbb{R}^d$, e.g. paraboloid or sphere, or curve
$(t, t^2, t^3, \cdots, t^d)$.
- Kakeya/Besicovitch/Nikodym maximal functions.

What does "underlying geometry" mean?

Alternative description: $\boxed{\textbf{Incidence manifold}}$ $\mathcal{I} \subset X \times X^\star$. Natural projections $\pi, \pi_\star$ from $\mathcal{I}$ to $X, X^\star$. Hypothesis: Both projections are submersions. $\boxed{\mathcal{I} = \{(x,y) : y \in M_x\}}$

Consider $\mathcal{V}, \mathcal{V}^\star =$ all vector fields tangent to the level sets of $\pi^\star, \pi$ respectively. $\mathcal{V}, \mathcal{V}^\star$ are separately integrable, but $\mathcal{V} + \mathcal{V}^\star$ need not be.

Under the submersion hypothesis, both T, T^* preserve L^p for all $1 \le p \le \infty$. T is said to be L^p-improving if there exists $q > p$ such that $T : L^p \to L^q$. (This property is independent of $p \in (1, \infty)$.)

The natural symmetry group here is

$$\mathrm{Diff}(\mathbb{R}^{\mathrm{d}}) \times \mathrm{Diff}(\mathbb{R}^{\mathrm{d}^\star}).$$

Nagel-Stein-Wainger-Christ proved:

T is L^p-improving $\Leftrightarrow V + V^\star$ satisfies the bracket condition.

(Part of their analysis is a sort of real analogue of the theory of Segre sets.) But the method produces a poor bound for q in terms of p in almost all cases. Obtaining the optimal range of exponents is a far subtler problem.

Two examples particularly relevant to this talk:

• Convolution with surface measure on paraboloid in $\mathbb{R}^d$:

$$Tf(x) = \int_{\mathbb{R}^{d-1}} f(x' - t, x_d - |t|^2)\, dt.$$

Maps $L^{(d+1)/d} \to L^{d+1}$.

• Convolution with arc measure on a curve:

$$Tf(x) = \int_{\mathbb{R}^1} f(x_1 - t, x_2 - t^2, x_3 - t^3, \cdots, x_d - t^d)\, dt.$$

Maps $L^p \to L^q$ for

$$p = \frac{d+1}{2} \text{ and } q = \frac{d+1}{2} \cdot \frac{d}{d-1}.$$

[Oberlin $d = 3, 4$]; [Christ (1997) $d > 4$]

For the first example there's a proof which relies on the basic $L^2 \mapsto H^{(d-1)/2}$ estimate for Fourier integral operators. For the second, the optimal gain within the scale of Sobolev spaces is insufficient and no proof along "Fourier/FIO" lines is known for $d \geq 4$.

Inequality

$$\|Tf\|_q \lesssim \|f\|_p$$

is essentially equivalent (by real interpolation) to

$$\mathcal{T}(E, E^\star) \lesssim |E|^{1/p} |E^\star|^{1/q'} \ \forall \text{ sets } E, E^\star$$

where

$$\mathcal{T}(E, E^\star) = \int_E T(\chi_{E^\star}) = \langle \chi_E, T(\chi_{E^\star}) \rangle.$$

Another way to put it:

$$|(E \times E^\star) \cap \mathcal{I}| \lesssim |E|^{1/p} |E^\star|^{1/q'}.$$

$|(E \times E^\star) \cap \mathcal{I}|$ is the (continuum) number of incidences between points in E and points in $E^\star$.

This is really a problem in (continuum) combinatorics.

Recall Szemerédi-Trotter theorem: N lines and N points in $\mathbb{R}^2$ have at most $CN^{4/3}$ incidences.

Here X = set of all points, $X^\star$ = set of all lines, and $\mathcal{I}$ = set of all pairs (p,ℓ) such that $p \in \ell$.

The S-T Theorem says

$$|(P \times L) \cap \mathcal{I}| \lesssim |P|^{2/3}|L|^{2/3}.$$

The combinatorial point of view rose to prominence in harmonic analysis through the work first of Bourgain, then of Wolff, and has since been developed further in fundamental works of Schlag, Katz, Tao, and others.

Quasi-extremals

Consider the operator T of convolution with surface measure on paraboloid in $\mathbb{R}^d$. This is one of the most fundamental examples in the subject. It enjoys a wealth of symmetry; via the action of $\mathrm{Diff}(\mathbb{R}^{\mathrm{d}}) \times \mathrm{Diff}(\mathbb{R}^{\mathrm{d}})$ it is equivalent to a certain left-invariant operator on the Heisenberg group of dimension $2d-1$.

The basic inequality is

$$\mathcal{T}(E, E^\star) \le C_0 |E|^{d/(d+1)} |E^\star|^{d/(d+1)}.$$

Definition: A pair of sets $(E, E^\star)$ is an ε-quasi-extremal if

$$\mathcal{T}(E, E^\star) \ge \varepsilon C_0 |E|^{d/(d+1)} |E^\star|^{d/(d+1)}.$$

To describe *all* ε-quasi-extremals is to refine the $L^p \mapsto L^q$ inequality. My hope is that an argument sufficiently fine to prove this must be robust, and hence potentially useful for more general cases.

Definition. For any
point $\bar{z} = (\bar{x}, \bar{x}_\star) \in \mathcal{I}$,
$\rho > 0$,
orthonormal basis $\mathbf{e} = \{e_1, \cdots, e_{d-1}\}$ for $\mathbb{R}^{d-1}$,
and any $r, r^\star \in (\mathbb{R}^+)^{d-1}$ satisfying

$$r_j r_j^\star = \rho \; \forall \, 1 \le j \le d-1$$

$\mathcal{B}(\bar{z}, \mathbf{e}, r, r^\star)$ denotes the set of all $z = (x, x_\star) \in \mathcal{I}$ satisfying all of

$$\begin{gathered} |\langle x' - \bar{x}', e_j \rangle| < r_j \;\; \forall \, j, \\ |\langle x'_\star - \bar{x}'_\star, e_j \rangle| < r_j^\star \;\; \forall \, j, \\ |x_d - \bar{x}_d - |x' - \bar{x}'|^2| < \rho, \\ |(x_\star)_d - (\bar{x}_\star)_d + |x'_\star - \bar{x}'_\star|^2| < \rho. \end{gathered}$$

$$\boxed{(B, B^\star) = (\pi\mathcal{B}, \pi_\star\mathcal{B})}$$

- These sets are tubular neighborhoods in $\mathbb{R}^d$ of graphs of an explicit quadratic polynomial over convex subsets of $\mathbb{R}^{d-1}$.
- They are paired up in a specific way.

Write $\mathcal{T}(E, E^\star) = \langle T(\chi_{E^\star}), \chi_E \rangle$.

Theorem. Let $d \geq 2$. There exist $C, A < \infty$ such that: $\forall\, \varepsilon > 0$ and $\forall$ ε-quasi-extremal pairs $E, E^\star \subset \mathbb{R}^d$, there exists a ball $\mathcal{B} \subset \mathcal{I}$, as defined above, such that the associated pair $(B, B^\star) = (\pi(\mathcal{B}), \pi^\star(\mathcal{B}))$ satisfies

$$\mathcal{T}(E \cap B, E^\star \cap B^\star) \geq C^{-1} \varepsilon^A \mathcal{T}(B, B^\star)$$

and

$$|B| \leq C|E| \text{ and } |B^\star| \leq C|E^\star|.$$

It follows that $|E \cap B| \geq C^{-1}\varepsilon^A |E|$ and $|E^\star \cap B^\star| \geq C^{-1}\varepsilon^A |E^\star|$.

The existence of such a **high-dimensional family of extremals** is due to two effects acting in concert.

- High degree of symmetry.

- Degeneracy. There exist pairs of manifolds $(Y, Y^\star)$ of dimensions $(k, d-1-k)$ such that

$$Y \times Y^\star \subset \mathcal{I}.$$

(Think of the Heisenberg group with the usual vector fields X_j, Y_j $(1 \le j \le d-1)$ and central vector field T. Then for any set $S \subset \{1, 2, \cdots, d-1\}$, $[X_j, Y_k] = 0$ whenever $j \in S$ and $k \notin S$.)

The natural generalization is a contact structure of dimension $2d-1$, equipped with two transverse foliations by $(d-1)$-dimensional leaves. Then such pairs $(Y, Y^\star)$ will typically not exist, and the family of quasi-extremals should be much smaller.

Consider the **general situation** $\mathcal{I} \subset X \times X^\star$, satisfying the bracket/L^p-improving condition.

Define

$$\Lambda(e, e_\star) = \sup_{|E|=e, |E^\star|=e_\star} \mathcal{T}(E, E^\star).$$

We say $\boxed{\textit{subalgebraic almost-extremals exist}}$ if $\forall\, \delta > 0$, $\forall\, e, e_\star \in (0, 1]$, there exist sets $E, E^\star$ of measures $e, e_\star$ such that

$$\mathcal{T}(E, E^\star) \geq c_\delta e^\delta e_\star^\delta \Lambda(e, e_\star)$$

and

$$E, E^\star \text{ are subalgebraic sets of degrees } \leq N,$$

where N is permitted to depend on δ but not on $e, e_\star$. (Likewise uniformly bounded *complexity*.)

The qualifier "almost" refers here to the sacrificed factor $e^\delta e_\star^\delta$.

Two conjectures.

• Subalgebraic almost-extremals exist, for all $\mathcal{T}$ satisfying the L^p-improving/bracket hypothesis. described above.
• Any ε-quasi-extremal pair has a large subalgebraic subpair.

This means: If $\mathcal{T}(E, E^\star) \geq \varepsilon \Lambda(|E|, |E^\star|)$ then there exist subalgebraic sets $\mathcal{E}, \mathcal{E}^\star$, of uniformly bounded degrees and complexities, whose measures are comparable to the measures of $E, E^\star$ respectively, such that

$$\mathcal{T}(E \cap \mathcal{E}, E^\star \cap \mathcal{E}^\star) \geq c\varepsilon^A (|E| \cdot |E^\star|)^\delta \mathcal{T}(E, E^\star).$$

Today's theorem is rather narrow, but illustrates the quite general conjectures. The argument I'll outline does succeed in isolating all quasi-extremals.

Young's convolution inequality

$$\left| \iint_{\mathbb{R}^2} f(x)g(y)h(x-y)\,dx\,dy \right| \le C\|f\|_p\|g\|_q\|h\|_r,$$

where $p^{-1} + q^{-1} + r^{-1} = 2$, illustrates the distinction between the above two conjectures.

Let $\delta > 0$ be small. Taking f, g, h to be intervals, centered at the origin, of some common length δ produces subalgebraic quasi-extremals.

But taking each function to be a δ-neighborhood of $\{n \in \mathbb{Z} : |n| \le N\}$ for arbitrarily large N, then rescaling so that the total measure is δ, produces equally optimal quasi-extremals, uniformly in N, δ so long as $\delta \le \frac{1}{4}$.

The complexity of these sets tends to infinity with N.

The point here is not the precise formulation of the conjecture, but rather its general spirit. Any statement roughly along these lines would be welcome.

For other problems like oscillatory integrals, one would like quasi-extremals to correspond to appropriate subalgebraic sets in phase space; but one should begin with simpler operators involving nonnegative quantities, as above.

Christ [1997] introduced the following essentially combinatorial method: Define the $\boxed{\textbf{average numbers of incidences}}$

$$\alpha = \mathcal{T}(E, E^\star)/|E| \qquad \alpha_\star = \mathcal{T}(E, E^\star)/|E^\star|.$$

Given a $\boxed{\text{lower bound}}$ on the number of incidences $\mathcal{T}(E, E^\star)$, seek $\boxed{\text{lower bounds}}$ on $|E|, |E^\star|$.

There exists $x_0 \in E$ with at least an average number of incidences; thus $\sigma(M_{x_0} \cap E^\star) \geq \alpha$. Thus we've located a family of points in $E^\star$, of some dimension k, with a lower bound on the k-dimensional measure.

Now *iterate*. If we're lucky, for most points in $x_\star \in M_{x_0} \cap E^\star$, the same reasoning produces a family of points in $E \cap M^\star_{x_\star}$. This gives rise to a subset of E of formal dimension $k + k_\star$.

Next iteration produces a subset of $E^\star$ of formal dimension $k + k_\star + k$, and so on.

This construction produces a set $\Omega \subset \mathbb{R}^D$ having certain structure and in particular, a lower bound on $|\Omega|$ in terms of $\alpha, \alpha_\star$, and a map $\Phi : \Omega \to E$, defined in terms of $\mathcal{I}$. Here D depends on the number of iterations and on the dimensions $k, k_\star$ of the manifolds $M_x, M^\star_y$.

Example:

$\phi(t_1, t_2, \cdots, t_d) =$
$(t_1 - t_2 + t_3 - \cdots, t_1^2 - t_2^2 + t_3^2 - \cdots, \cdots, t_1^d - t_2^d + t_3^d - \cdots)$

arises for convolution with arc measure on $(t, t^2, \cdots, t^d)$.

In the $\boxed{\textbf{equidimensional case}}$ $D = d$,

$$|E| \geq |\Phi(\Omega)| \geq c \int_\Omega |J|$$

where J is the Jacobian determinant of Φ. The geometry is now encoded by the degree to which J vanishes. The bracket condition means that it vanishes to finite order, so we're essentially in the realm of algebraic degeneracy.

Alas

- Argument sketched on preceding slide doesn't give the correct bound for $(t, t^2, t^3, \cdots)$.
- For most combinations of dimensions (ambient and submanifolds), this procedure doesn't lead to an equidimensional mapping.

Christ [1997] completed the analysis for that case by iterating $2d-2$ times, rather than the natural number d, and implementing an *ad hoc* argument relying on explicit properties of Vandermonde determinants.

A very simple but suggestive fact: Given a polynomial J, and given $\delta > 0$, consider the minimum value of $\int_S |J|$, taken over all sets S of measure δ.

For this subproblem,

$\boxed{\text{optimal sets } S \text{ are obviously subalgebraic.}}$

Tao and Wright [2003] turned this method into a general theory for the case when M_x and $M^\star_{x^\star}$ are all *one-dimensional*, by (i) reinterpreting the construction, and (ii) introducing a key conceptual idea.

Their theorem essentially says that all quasi-extremals have large intersection with two- parameter Carnot-Caratheodory balls $B(z, \delta, \delta_\star)$ in $\mathcal{I}$.

$(\alpha, \alpha_\star)$ can be interpreted as the $\boxed{\text{bi-radius}}$ $(\delta, \delta_\star)$ of such a ball.

The process of constructing successively higher-dimensional subsets of $E, E^\star$ is interpreted as flowing alternately along vector fields $V, V^\star$ in $\mathcal{I}$ tangent to the foliations defined by $\pi, \pi_\star$.

However, this is no ordinary flow . . . $\boxed{\text{It jumps}}$.

What is the analogue of Carnot-Caratheodory balls, when the flow is with respect to a **multi-dimensional time**?

For 1D time, these balls are the maximal region which can be visited by a traveler who has a given amount of time.

In the general case, they seem to correspond roughly to a minimal region visited by a reluctant traveler who would rather stay home, but who must expend a fixed amount of fuel and has only the option of choosing the slowest, most fuel-inefficient roads.

Today's theorem demonstrates that such "balls" (which are merely conjectural in the general geometric setup) are not uniquely specified by their center and the "bi-time" $(\alpha, \alpha_\star)$ — in contrast to the 1D case.

Proof of Theorem on Quasi-Extremals

Step 1 Given $E, E^\star$, there exist $x_0 \in \mathbb{R}^{d-1}$ and

$$\Omega_1 \subset \mathbb{R}^{d-1}, \quad \Omega_2 \subset \mathbb{R}^{d-1} \times \mathbb{R}^{d-1}$$

such that

- $|\Omega_1| \sim \alpha$,
- Ω_2 is fibered over Ω_1
  (which means: $(s,t) \in \Omega_2 \Rightarrow s \in \Omega_1$),
- All fibers have measures $\sim \alpha_\star$.
- $\Phi(\Omega_2) \subset E$

where $\Phi(s,t) = x_0 - (s, |s|^2) + (t, |t|^2)$.

Step 2 — Inflation

Consider • $\Omega^\dagger = \{(s, t_1, \cdots, t_{d-1}) : (s, t_j) \in \Omega_2 \, \forall j\}$.
• $\Psi(s, t_1, \cdots, t_{d-1}) = \big(\Phi(s, t_1), \Phi(s, t_2), \cdots, \Phi(s, t_{d-1})\big)$.
Then $\Psi(\Omega^\dagger) \subset E^{d-1}$.

This is an $\boxed{\text{equidimensional}}$ situation, so

$$|E|^{d-1} \gtrsim \int_{\Omega^\dagger} |J|$$

where J is the Jacobian determinant of Ψ;

$$|J| = |\det(t_1 - s, t_2 - s, \cdots, t_{d-1} - s)|.$$

It's easy to show that

$$\int_{\Omega^\dagger} |J| \gtrsim \alpha \alpha_\star^d,$$

which directly gives the optimal power-law bound

$$|\mathcal{T}(E, E^\star)| \lesssim |E|^{d/d+1} |E^\star|^{d/d+1}.$$

Moreover, one can show that $\boxed{\text{either}}$
(i) For most s, the fiber over s in Ω_2 intersects some (balanced) $\boxed{\text{convex}}$ set (centered at s) of measure $\leq C\varepsilon^{-C}\alpha_\star$, or
(ii) The power-law bound can be so significantly improved that $(E, E^\star)$ is not an ε-quasi-extremal pair.

Step 3 Any subset of Euclidean space can be well approximated by a convex set (**!**).

Lemma For any dimension $n \geq 1$ and any small $\eta > 0$, there exist $\delta, c > 0$ with the following property.

For any set $S \subset \mathbb{R}^n$ satisfying $0 < |S| < \infty$,

there exists a bounded convex set $\mathcal{C} \subset \mathbb{R}^n$ so that

$$|\mathcal{C}| \geq c_\eta |S|$$

and for any convex set $\mathcal{C}' \subset \mathcal{C}$,

$$|\mathcal{C}'| \leq \delta |\mathcal{C}| \Rightarrow |S \cap (\mathcal{C} \setminus \mathcal{C}')| \geq c(|S|/|\mathcal{C}|)^\eta |S|.$$

• Heuristically, $S \cap \mathcal{C}$ is a generic, diffuse subset of $\mathcal{C}$.
• The factor $(|S|/|\mathcal{C}|)^\eta$ represents a slight loss when $|\mathcal{C}| \gg |S|$.

A key part of the idea of Tao and Wright was to bring in a variant of this lemma in dimension one.

Erdogan-C sharpened the 1D case as above.

Step 4

By interchanging the roles of $E, E^{\star}$ and feeding this information back in, we conclude that:

For any ε-quasi-extremal pair, the parameter set Ω_1 can be taken to be a generic subset of some convex set $\mathcal{C}$, in the sense of Step 3,

satisfying

$$|\mathcal{C}| \lesssim \varepsilon^{-C}\alpha.$$

Step 5 — Slicing

For each affine line $L \subset \mathbb{R}^{d-1}$, consider $\Phi_L(s,t) = \Phi(s,t)$ only for $s \in L \cap \Omega_1$, and $(s,t) \in \Omega_2$.

Trivially the range of Φ_L is contained in E.

The domain now has dimension d, so this is an equidimensional situation. This gives a lower bound for $|E|$, by integrating the Jacobian of Φ_L.

Averaging this bound over all L (taking $\mathcal{C}$ into account) gives

$$|E| \geq |\Phi(\Omega_2)| \gtrsim |\det(A)|^{-1} \int_{\Omega_2} |A(t-s)|\, ds\, dt$$

where A is a symmetric linear transformation chosen so that $\mathcal{C}$ is comparable to the ellipsoid A(unit ball).

The variable s is already constrained to a convex set; this integral is large unless $t - s$ is constrained to a corresponding (dual) convex set.

Working out the consequences gives the theorem.