

REDUCTION THEOREM FOR GENERALIZED INDISCERNIBLES IN STABLE THEORIES

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ABSTRACT. We show that for *companionable* T , and stable T' , any T -indiscernible in T' remains indiscernible with respect to the *fully stable* formulas in T .

1. INTRODUCTION

In the classical example of a characterization by indiscernibles we have from Shelah [2] that a theory T is stable just in case any indiscernible sequence in a model of T is an indiscernible set. Let I be the countable dense linear order, the former is equivalent to the claim that every I -indiscernible in a model of T is an $\{=\}$ -indiscernible. We can interpret this as a statement like, “every I -indiscernible remains indiscernible with respect to Δ where Δ is exactly the formulas which are stable under any partition of their variables to “ \bar{x} ” and “ \bar{y} ”. In fact, we will call Δ the *fully stable* formulas of the language of I .

We prove a claim that is closely related to the classical example. We impose a hypothesis on T that would be satisfied if T were the model companion of a universal theory in a relational language. We call this hypothesis companionability. For the following, assume f_m is a function that maps τ to a conjunction of m instances of τ and its negation, with possibly overlapping variables. (e.g. a finite fragment of the order property for τ .)

Definition 2.4. Let $f_m(\tau(x)) =: \Phi(x_0, \dots, x_l)$, for some f_m . We say T is companionable if

- T is \aleph_0 -categorical
- for any finite list of parameters \bar{c} , for any non-algebraic $p(x; \bar{c})$ compatible with $\tau(x)$ or $\neg\tau(x)$, for any \bar{a} satisfying $\Phi(\bar{x})$, there is a \bar{b} satisfying $\Phi(\bar{x}) \wedge \bigwedge_{i \in I} p(x_i; \bar{c})$.

We will also want a generalization on the definitions from [1]:

Definition 2.2. By a T -indiscernible in T' a sequence of parameters $(a_i : i \in J)$ indexed by some \aleph_0 -saturated model $J \models T$ where the parameters come from some model $M \models T'$ such that this sequence is J -indiscernible in M .

Say the *fully-stable* formulas of T are those formulas $\varphi(\bar{x}, \bar{y})$ that are stable under any allocation of their variables to \bar{x}, \bar{y} . Then we have the following theorem:

Theorem 1.1. *For companionable T , and stable T' , any T -indiscernible in T' remains indiscernible with respect to the fully stable formulas in T .*

If we assume that T is unstable, then we get a partial converse to the theorem. If T' is not stable, then there exists a T -indiscernible in T' that is not indiscernible with respect to the *fully stable* formulas in T .

It is the author's hope that the companionability assumption on T can be eliminated if we make the conclusion on the T -indiscernible slightly weaker. For instance, we could ask the T -indiscernible $(a_i : i \in J)$ for $J \models T$ to remain indiscernible with respect to not just the fully stable formulas in T , but also with respect to formulas $\varphi(\bar{x}; \bar{a}_i)$ that are fully-stable – in other words we allow parameters from the sequence. We would also need a methodology to grow T -indiscernibles, since we would need to take our original set of parameters and grow them to an \aleph_0 -saturated index set in some elementary extension of our model $M \models T'$, while still retaining the important parts of the type structure in the model J of T . This is necessary because an unstable formula in T may not exhibit the full order-property in arbitrary $J \models T$. This is probably a good method to develop anyway.

Now we prove the theorem.

2. CHARACTERIZATION OF STABLE T BY GENERALIZED INDISCRIBIBLES

In [1] we have the following definition of generalized indiscernible

Definition 2.1. Let Δ be some subset of L and Δ' some subset of L' . For any L -structure I and L' -structure M we say that

- (1) a sequence $(a_i : i \in I)$ is (Δ, Δ') -*indiscernible* in M if for any finite n , for any tuples $i_1, \dots, i_n, j_1 \dots j_n$ from I ,

$$i_1, \dots, i_n \equiv_{\Delta}^I j_1 \dots j_n \Rightarrow a_{i_1}, \dots, a_{i_n} \equiv_{\Delta'}^M a_{j_1}, \dots, a_{j_n}$$

- (2) a sequence $(a_i : i \in I)$ is Δ -*indiscernible* in M if it is (Δ, L') -indiscernible in M

We will refer to any (L, L') -indiscernible $(a_i : i \in I)$ as an I -*indiscernible* in M

In this paper, we will need an expanded definition of generalized indiscernible.

Definition 2.2. By a T -*indiscernible* in T' we mean a sequence of parameters $(a_i : i \in J)$ indexed by some \aleph_0 -saturated model $J \models T$ where the parameters come from some model $M \models T'$ such that this sequence is J -indiscernible in M .

We will also make use of the notion of a *fully-stable* formula.

Definition 2.3. A formula $\varphi(x_0, \dots, x_{n-1})$ in a 1-sorted language L is *fully-stable* if for any subset $w \subsetneq n$ of size m and orderings on $w = \langle w_0, \dots, w_{m-1} \rangle$ and $n \setminus w = \langle v_m, \dots, v_{n-1} \rangle$ the formula $\varphi(x_{w_0}, \dots, x_{w_{m-1}}; x_{v_m}, \dots, x_{v_{n-1}})$ is stable.

In the following, let f_m be some finite conjunction of $+/- \tau$ in some finite list of possibly overlapping variables, e.g. for $w \subset n$, $f_m = \bigwedge_{i < n} \tau(x_i)^{i \in w}$

Definition 2.4. Let $f_m(\tau(x)) =: \Phi(x_0, \dots, x_l)$, for some f_m . We say T is *companionable* if

- T is \aleph_0 -categorical
- for any finite list of parameters \bar{c} , for any non-algebraic $p(x; \bar{c})$ compatible with $\tau(x)$ or $\neg\tau(x)$, for any \bar{a} satisfying $\Phi(\bar{x})$, there is a \bar{b} satisfying $\Phi(\bar{x}) \wedge \bigwedge_{i \in l} p(x_i; \bar{c})$.

Here is a stronger condition than 2: suppose we are given d_i and \bar{c} , each $\text{tp}(d_i/\bar{c})$ non-algebraic and compatible with $\text{tp}(a_i)$, for all i . There are b_i ($i \leq n$) where

- (1) $b_i \equiv a_i$
- (2) $b_i \equiv_{\bar{c}} d_i$

In the above, we would take each a_i to be a pair c, d satisfying either $\tau(c, d)$ or $\neg\tau(c, d)$.

Theorem 1.1 (Stable reduction theorem). *For companionable T , and stable T' , any T -indiscernible in T' remains indiscernible with respect to the fully stable formulas in T .*

Proof. For the forward direction, see Lemma 2.5. For the backward direction Suppose T' were unstable, then there would be a formula in T' with the order property. There is also such a formula in T . For any finite n , look at the order property configuration in T , and pick representatives of the n -parameters. Supposedly we can embed this into the formula from T' ? (problem is if we are going from many variables in T to one variable in T' .)

Another problem with the backward direction, what if Δ contains orders of length n defined by φ_n . Won't this give rise to ordered θ if we map them all into θ , for successive indiscernibles? So basically the fully-stable part cannot have increasing orders by formulas (is this a second order statement?) It actually seems like an infinite disjunction of formulas. \square

Lemma 2.5. *Here we prove that if T' is stable, then any T -indiscernible in T' remains indiscernible with respect to the fully-stable formulas of T .*

Proof. Suppose we have an I -indiscernible that fails to be indiscernible with respect to the fully-stable formulas of T . We want to show that T' is not stable by producing a T' -formula with the order property. Let Δ be the fully-stable formulas of T . By Δ -type in the following, I am going to mean type according to $+/-$ of formulas in Δ , not allowing other operations to build new formulas.

By assumption there is some n where the Δ -indiscernibility fails. In other words, we have \bar{i} and \bar{j} of length n in some model I of T that have the same Δ -type, but fail to map to the same type in M , a model of T' . Therefore, there is some formula θ in T' where they disagree.

Let i_0, j_0 be our starting \bar{i} and \bar{j} (we're going to drop the tuple notation) and let p_0 be a conjunction of all the formulas that they agree on (this includes all the n -ary fully stably formulas, by assumption.)

(Induction Hypothesis): At stage $m+1$. We are given a $p_m(x_1, \dots, x_n)$ and i_m, j_m that satisfy p_m and agree on all the fully-stable formulas, but map to different values of θ . Let τ_m be the first formula in L' that has not been dealt with.

- case 1:** There is an i_{m+1}, j_{m+1} that satisfy $p_m \wedge \tau_m$ and agree on all the fully-stable formulas, but map to different values of θ . Then let p_{m+1} be $p_m \wedge \tau_m$ and go to the next stage. .
- case 2:** There is an i_{m+1}, j_{m+1} that satisfy $p_m \wedge \neg\tau_m$ and agree on all the fully-stable formulas, but map to different values of θ . Then let p_{m+1} be $p_m \wedge \neg\tau_m$ and go to the next stage.
- case 3:** Actually, both are constant. In this case we know $p_m \wedge \tau_m$ implies θ and $p_m \wedge \neg\tau_m$ implies $\neg\theta$. (wlog) We know that τ_m is not fully stable because, in the previous stage, we were able to find two i_m, j_m agreeing on all the fully-stable formulas but not τ_m . Since τ_m is not fully stable, flipping τ_m over p_m witnesses that θ is unstable in T' .

Note that, because of the procedure, $p_m(\bar{x}, \bar{y})$ where \bar{y} are the variables not in τ , must isolated the type of \bar{y} (no assumptions necessary on the theory. If \bar{y} parts had disagreed on formulas they would have been unstable formulas and dealt with previously. Also, it can't be τ_m . \square

REFERENCES

- [1] L. SCOW, *Characterization of NIP theories by random ordered graph-indiscernibles*, (in preparation)
- [2] S. SHELAH, *Classification Theory*, Name of series, Springer-Verlag, 1978.