

CHARACTERIZATION OF NIP THEORIES BY RANDOM ORDERED GRAPH-INDISCERNIBLES

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ABSTRACT. We show that T is NIP if and only if every random ordered graph-indiscernible in a model of T is strictly order-indiscernible.

1. INTRODUCTION

We know from [6] the classical result that a theory T is stable if and only if for any model M of T , every indiscernible sequence in M is an indiscernible set. Say that a sequence indexed by a structure I is a *generalized indiscernible in M* if any tuples of indices having the same quantifier-free type in I map to tuples of elements having the same type in M . (see [2, 1] for previous examples of generalized indiscernibles) If we examine the classical result more closely, we see that it characterizes a combinatorial weakness-property of T (i.e., stability) by way of the following: that any sequence in a sufficiently-saturated model of T indiscernible with respect to the language L with signature $\{<, =\}$ is indiscernible strictly with respect to the language of pure equality (signature $\{=\}$). This paper aims to generalize on this idea. We characterize combinatorial weakness-properties of theories T in terms of whether, for some fixed reduct language L_0 of the indexing model I , any I -indiscernibles in a sufficiently-saturated model $M \models T$ remains indiscernible with respect to the reduct $I \upharpoonright L_0$.

We prove the following characterization of NIP theories, where \mathcal{R} is the random ordered graph:

Theorem 1.1. *A theory T has the independence property if and only there exists an \mathcal{R} -indiscernible in some model of T that is not strictly order-indiscernible.*

In other words, a theory T is NIP just in case any \mathcal{R} -indiscernible in a model of T is strictly order-indiscernible.

2. CHARACTERIZATION OF NIP T

Usually all tuples \bar{a} are assumed to be finite and all models M used are countably saturated; models M are L' -structures with theory T' and index models I are L -structures with theory T .

Definition 2.1. For M an L -structure, $\Delta(x_1, \dots, x_m)$ some set of L -formulas closed under permutations of variables and two length n tuples \bar{a}, \bar{b} from M , by

$$a_1, \dots, a_n \equiv_{\Delta}^M b_1, \dots, b_n$$

we mean that $\varphi(a_{i_1}, \dots, a_{i_m}) \leftrightarrow \varphi(b_{i_1}, \dots, b_{i_m})$ for all increasing i_1, \dots, i_m from $\{1, \dots, n\}$ and $\varphi(x_1, \dots, x_m)$ from $\Delta(x_1, \dots, x_m)$.

Definition 2.2. Let Δ be some subset of L and Δ' some subset of L' . For any L -structure I and L' -structure M we say that

- (1) a sequence $(a_i : i \in I)$ is (Δ, Δ') -indiscernible in M if for any finite n , for any tuples $i_1, \dots, i_n, j_1 \dots j_n$ from I ,

$$i_1, \dots, i_n \equiv_{\Delta}^I j_1 \dots j_n \Rightarrow a_{i_1}, \dots, a_{i_n} \equiv_{\Delta'}^M a_{j_1}, \dots, a_{j_n}$$

- (2) a sequence $(a_i : i \in I)$ is Δ -indiscernible in M if it is (Δ, L') -indiscernible in M

We will refer to any (L, L') -indiscernible $(a_i : i \in I)$ as an I -indiscernible in M

Consider the language for ordered graphs, $L = \{R, <\}$, for binary $R, <$. Then let T_0 be the theory for ordered (symmetric) graphs: no point is R -related to itself; R is symmetric; $<$ is a linear order.

We intend our index structure to model the theory T axiomatized below. We'll use the following notation: let the q_n^i for $1 \leq i \leq n!$ represent all the order types on n variables.

[given some order type over the x's and y's, $p(z;x,y)$ there is z realizing it connected to x's and disconnected from y's. not enough to just make the two $n+1$ -types consistent...would fail to realize some types where the x's interact with the y's....]

- (1) T_0
(2) $(n < \omega, i, j \in \{1, \dots, n!\}, k, l \in \{1, \dots, (n+1)!\}, q_n^i \wedge q_{n+1}^k$ satisfiable and $q_n^j \wedge q_{n+1}^l$ satisfiable):

$$(\forall x_1 \dots x_n, y_1, \dots, y_n) (\bigwedge_{i,j} x_i \neq x_j \wedge \bigwedge_{i,j} y_i \neq y_j \wedge q_n^i(\bar{x}) \wedge q_n^j(\bar{y}) \wedge \bigwedge_{i,j} x_i \neq y_j \rightarrow (\exists z) [q_{n+1}^k(\bar{x}; z) \wedge q_{n+1}^l(\bar{y}; z) \wedge \bigwedge_{i \leq n} (zRx_i \wedge \neg zRy_i)])$$

Claim 2.3. T' is the model completion of T , T' is \aleph_0 -categorical and eliminates quantifiers. Any n -type in T' is the product of 2-types.

[fix...better proof]

Proof. Note that T and T' are co-theories and T eliminates quantifiers, by a back-and-forth argument between countable models of T . Thus T is the model companion

of T_0 . The back and forth argument shows that countable models of T are ultrahomogeneous. T has a finite relational language and eliminates quantifiers, thus is \aleph_0 -categorical. \square

By \aleph_0 -categoricity, we may refer to the countable model $\mathcal{R} \models T$ as the *random ordered graph*.

Remark 2.4. Note that \mathcal{R} is densely ordered. Moreover, graph types are dense in the order over finite sets of parameters: for any R -type $p(\bar{x}; \bar{a})$ and order-type $q(\bar{x}; \bar{a})$, we have that $p(\bar{x}; \bar{a}) \cap q(\bar{x}; \bar{a})$ is infinite.

Remark 2.5. It is easy to see that T has the independence property. In fact, this is witnessed by xRy in either order type: $x < y$ or $y < x$.

To build our generalized indiscernible we will use the notion of a Ramsey Class from [4]. We use the notation $\binom{B}{A}$ to denote the A -substructures of B by which we mean the embeddings of A into B . We use the more model-theoretic notation in [3] over the category-theoretic presentation in [4].

Definition 2.6 (Nešetřil [3]). A class \mathcal{K} of finite L -structures has the A -Ramsey property if for every integer $k > 0$ and $B \in \mathcal{K}$ there exists $C \in \mathcal{K}$ such that $C \rightarrow \binom{B}{k}^A$, where the previous means that for every partition of $\binom{C}{A} = a_1 \cup \dots \cup a_k$ there exists $B' \in \binom{C}{B}$ and $i \in \{1, \dots, k\}$ such that $\binom{B'}{A} \subset a_i$.

A class \mathcal{K} is known as a *Ramsey Class* if it has the A -partition property for all $A \in \mathcal{K}$.

Let L be a first order language with signature Δ , \mathcal{K} be a class of finite L -structures and \mathcal{F} some subclass of irreducible structures F , meaning any two elements are part of some tuple that satisfies some relation in Δ . Let $\text{Forb}_\Delta(\mathcal{F})$ denote the class of structures in \mathcal{K} that do not embed any $F \in \mathcal{F}$ as a substructure; let $\overrightarrow{\text{Forb}}_\Delta(\mathcal{F})$ be all the members of $\text{Forb}_\Delta(\mathcal{F})$, where each finite structure is given with a linear order [may think of members as expansions to the language with order]. We know from [5]:

Theorem 2.7 (Nešetřil-Rödl). *The class $\overrightarrow{\text{Forb}}_\Delta(\mathcal{F})$ is Ramsey for every type Δ .*

$\text{Age}(\mathcal{R})$ is a Ramsey Class according to [3] where we take Δ to contain the binary relation $\{R\}$ and assume \mathcal{K} to be composed of the structures:

- $xRy \wedge \neg yRx$
- xRx

Since $\text{Age}(\mathcal{R})$ is a Ramsey Class, we have the existence of generalized \mathcal{R} -indiscernibles in all models. Moreover, these indiscernibles have the *modeling property*: For any $(\bar{a}_i : i \in \mathcal{R})$ in some model M there exist $(\bar{b}_i : i \in \mathcal{R})$ that are \mathcal{R} -indiscernible in M but also are modeled on the \bar{a} 's – for every finite subset of L' -formulas Δ , every Δ -type of a sequence of \bar{b} 's is identical to the Δ -type of some isomorphic-in- \mathcal{R} sequence of \bar{a} 's.

Theorem 1.1 (NIP Characterization Theorem). *A theory T has the independence property if and only there exists an \mathcal{R} -indiscernible in some model of T that is not strictly order-indiscernible.*

The proof is by Lemmas 2.8 and 2.9.

We'll be using the following notation: for $\bar{i} = (i_1, \dots, i_n)$ some tuple of indices, by $\bar{a}_{\bar{i}}$ we mean $(a_{i_1}, \dots, a_{i_n})$.

Lemma 2.8 (\Leftarrow). *Suppose T has the independence property. Then there exists an \mathcal{R} -indiscernible in some model of T which fails to be strictly order-indiscernible.*

Proof. Let T be as above and choose $M \models T$ to be some \aleph_0 -saturated model of T . By Lemma 2.2 of [2], there exists a formula $\varphi(\bar{x}; \bar{y})$ that codes graphs in T . In other words, φ embeds every finite graph relation. Let $G \subset |M|^2$ be the graph of φ in M . Since M is countably saturated, G contains the relation for an infinite random graph, thus embeds $\mathcal{R}|\{R\}$, since this is the prime model for random graphs. Let S be the image of \mathcal{R} under this embedding (push forward the order, $<$).

We claim a realization in M of the following type gives an \mathcal{R} -indiscernible that is not order indiscernible:

[is there infinite N-R (perhaps with E-Rado) that would allow finding indiscernibles in a larger language? i.e. more colors]

$$\Gamma = (A) (\{\varphi(\bar{c}_i, \bar{c}_j) : iRj\} \cup \{-\varphi(\bar{c}_i, \bar{c}_j) : \neg iRj\}) \cup \\ (B) \{\theta(\bar{c}_{\bar{i}}) \leftrightarrow \theta(\bar{c}_{\bar{j}}) : \bar{i} \equiv \bar{j}; \bar{i}, \bar{j} \in \mathcal{R}; \theta \in L'\}$$

We will show this type is finitely satisfiable in M by showing part (B) is finitely satisfiable in S .

Let $F \subset \Gamma$ be finite. Then the formulas from L' that occur in (B) are $\theta_1, \dots, \theta_n$, and the tuples from \mathcal{R} in both (A) and (B) have some maximal length m . Given an m -type \bar{i}_0 , how do we get all isomorphic \bar{j} to agree on the θ_k ? Apply the partition property where $A := \bar{i}_0$ and B is a set large enough to contain instances of all m -types. You get C which is finite but big enough so that for any 2^n -coloring of m -tuples in C , there is a subset of C isomorphic to B wherein all m -tuples isomorphic to A get colored the same color. In this way, we can find a large set B' containing all possible m -types wherein we have indiscernibility as far as types isomorphic to \bar{i}_0 go. We need to repeat this process for as many times as there are m -types in \mathcal{R} . Each time we must pick B' with the additional consideration that it needs to embed the C we need at the subsequent step. [have forgotten to show that you can make all pairs increasing....] [why again to products of A not work?] b[write out more clearly....]

At the end of this process we have found a witness to F .

The realization of Γ is an \mathcal{R} -indiscernible in M that is not strictly order-indiscernible in M . By the axioms in T we can find i_1, i_2, j_1, j_2 in \mathcal{R} such that

- $i_1 < i_2 \wedge i_1 R i_2$, and
- $j_1 < j_2 \wedge \neg j_1 R j_2$

(this is by independence of the order over R). In this case

- $\varphi(\bar{c}_{i_1}, \bar{c}_{i_2})$, but
- $\neg\varphi(\bar{c}_{i_1}, \bar{c}_{i_2})$

showing that order does not determine type in M . □

Lemma 2.9 (\Rightarrow). *Suppose some \mathcal{R} -indiscernible in a model $M \models T$ fails to be strictly order-indiscernible. Then T has the independence property.*

Proof. Let T, M be as above and $J = (\bar{a}_i : i \in \mathcal{R})$ some \mathcal{R} -indiscernible in M . We may assume that M is \aleph_0 -saturated by taking an elementary extension. Suppose for contradiction that J fails to be strictly order-indiscernible. We will show that T is forced to have the independence property.

We've assumed that for some n and n -type in L , $p = \{x_1 < \dots < x_n\}$, two n -tuples \bar{i}, \bar{j} satisfying p map to $\bar{a}_{\bar{i}}, \bar{a}_{\bar{j}}$ in M with different $\theta(\bar{x}_1, \dots, \bar{x}_n)$ -types for some θ .

We may assume that the types of \bar{i}, \bar{j} are identical but for disagreement in the R truth value between the first two variables: $i_1 R i_2$ in \bar{i} and $\neg j_1 R j_2$ in \bar{j} .

We can assume this for two reasons:

- (1) Consider all the R types in n variables that have the order type $p(x_1, \dots, x_n)$. If we enumerate the unordered pairs from n we can identify each R -type with a tuple from $\prod_{\binom{n}{2}} Z/2Z$. The group $G = \prod_{\binom{n}{2}} Z/2Z$ acts on this set transitively.

Let A be the set of R -types in p that map to θ , B the set of types that map to $\neg\theta$. By failure of order-indiscernibility, each of A and B contain an element, say a, b respectively. By transitivity, some group element g takes a to b . This element g is a product $h_k \cdots \cdots h_1$ of basis vectors of the form (δ_i^j) , $j \in \binom{n}{2}$ where $\delta_i^j = 1 \leftrightarrow i = j$. There is some least m such that $h_m \cdots \cdots h_1 \cdot a$ fails to be in A .

Thus we can replace \bar{i}, \bar{j} with realizations in p of the types represented by $h_{m-1} \cdots \cdots h_1 \cdot a, h_m \cdots \cdots h_1 \cdot a$, respectively.

- (2) Meanwhile, the cases where the disagreement in R -type happens non-sequentially or not at the beginning of the string are similar to the case where the R truth value of $(1, 2)$ is permuted. Without loss of generality, we can assume the disagreement in R -type happens at the pair $(1, 2)$. Note at every place in the following argument $(1, 2)$ might as well have been (i, j) for any indices i and j .

Now that we have made these assumptions, note we may assume that the j_k equal the i_k for $k \geq 3$. This is because $\text{tp}(i_1, i_3, \dots, i_n) = \text{tp}(j_1, j_3, \dots, j_n)$, so we can replace j_1 by a conjugate of i_1 over i_3, \dots, i_n, j_1^* . Now find j_2^* such that $j_1 j_2 \equiv j_1^* j_2^*$ and $j_2^*, i_3, \dots, i_n \equiv j_2, j_3, \dots, j_n$. This is easy according to the axioms of T .

Since T eliminates quantifiers and the signature of L contains only binary relations, n -types in \mathcal{R} are the product of 2-types. Thus we have $j_1^*, j_2^*, i_3, \dots, i_n \equiv j_1, j_2, j_3, \dots, j_n$ and we can replace the latter by the former.

By \mathcal{R} -indiscernibility, whenever $x < y < i_3$ we have

$$xRy \Leftrightarrow \theta(x; y, a_{i_3}, \dots, a_{i_n})$$

as long as x relates to the i_3, \dots, i_n exactly as i_1/j_1 does and y relates to the i_3, \dots, i_n exactly as i_2/j_2 does. Thus it remains to embed an arbitrary graph of size m below a_3 with $P \subset \text{tp}(i_1/i_3, \dots, i_n)$ and $Q \subset \text{tp}(i_2/i_3, \dots, i_n)$.

Assuming we can do the above, then $\theta(x; y, a_{i_3}, \dots, a_{i_n})$ embeds any finite graph and so $\exists z_3, \dots, z_n \theta(x; y, z_3, \dots, z_n)$ has the independence property.

However to do the above is simply to embed the graph in m stages, at each stage appealing to the axioms of T .

□

REFERENCES

- [1] M. DŽAMONJA AND S. SHELAH, *On \triangleleft^* -maximality*, *Annals of Pure and Applied Logic*, vol. 125 (2004), no. 1-3, pp. 119-158
- [2] M. C. LASKOWSKI AND S. SHELAH, *Karp complexity and classes with the independence property*, *Annals of Pure and Applied Logic*, vol. 120 (2003), pp. 263-283
- [3] J. NEŠETRIL AND V. RÖDL, *Partitions of Finite Relational and Set Systems*, *Journal of Combinatorial Theory (A)*, vol. 22 (1977), pp. 289-312. fixx
- [4] J. NEŠETRIL AND V. RÖDL, *Partitions of Finite Relational and Set Systems*, *Journal of Combinatorial Theory (A)*, vol. 22 (1977), pp. 289-312.
- [5] J. NEŠETRIL AND V. RÖDL, *Partitions of Finite Relational and Set Systems*, *Journal of Combinatorial Theory (A)*, vol. 22 (1977), pp. 289-312. fixx
- [6] S. SHELAH, *Classification Theory*, Name of series, Springer-Verlag, 1978.