AN EXPOSITION OF THE ARGUMENT FOR FBTI'S IN "ON $\lhd^*$-MAXIMALITY"

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1. Introduction

Let M be any L_M-model and $T := {}^{\omega>}2$ the full binary tree on ω. Consider the language L_T that consists of a constant for the empty sequence and two 4-ary relations: (1) $x\wedge y \lhd z\wedge w$; (2) $(x\wedge y)^\frown\langle 0\rangle \lhd z\wedge w$.

Define a sequence of parameters $\langle \overline{a}_\eta : \eta \in T\rangle$, to be 1-fbti if for any two finite tuples $\overline{\eta}$, $\overline{\nu}$ that have the same quantifier-free type in T, $\overline{a}_{\overline{\eta}}$ and $\overline{a}_{\overline{\nu}}$ have the same type in M. We say that the sequence is 2-fbti if we ignore (2) in the language. (There is also an argument for how to get 2-fbti, but I'll just be talking about the first argument.)

In [1] Džamonja and Shelah prove that for a given sequence of parameters $\langle \overline{b}_\eta : \eta \in {}^{\omega>}2\rangle$ we can find 1-fbti $\langle \overline{a}_\eta : \eta \in {}^{\omega>}2\rangle$ "modeled" on the original set of parameters: i.e. for any tuple $\overline{\eta}$ from T and finite set Δ of formulas of L_M there exists a tuple $\overline{\nu} \approx \overline{\eta}$ such that $\overline{a}_{\overline{\eta}} \equiv_\Delta \overline{b}_{\overline{\nu}}$. This argument occurs in pages 30-35 of [1], and to the tree-theory novice, may seem fairly condensed. In the following, I will make explicit a few more of the details of how this result follows from the Halpern-Laüchli Theorem.

2. Definitions

Definitions of some terms:

$L :=$ some language
$M :=$ any L-model
$T := {}^{\omega>}2$ the full binary tree on ω
$L_T :=$ the language with signature:
(1) a constant denoting the empty sequence;
(2) a relation interpreted as $x \wedge y \lhd z \wedge w$;
(3) a relation interpreted as $(x \wedge y)^\frown\langle 0\rangle \lhd z \wedge w$

Remark 1. *For $\overline{i} = (i_1, \ldots, i_n)$ some tuple of indices, by $\overline{a}_{\overline{i}}$ we mean $(a_{i_1}, \ldots, a_{i_n})$ some tuples of elements.*

Definition 2. *A sequence of parameters from M, $\langle \overline{a}_\eta : \eta \in T\rangle$ is* 1-fbti *if for any two finite tuples $\overline{\eta}$, $\overline{\nu}$ that have the same quantifier-free type* *in L_T (abbreviate this condition by "$\overline{\eta} \sim_1 \overline{\nu}$"), $\overline{a}_{\overline{\eta}}$ and $\overline{a}_{\overline{\nu}}$ have the same type in M.*

Claim 2.11 (in [1], Džamonja and Shelah)**.** *Given a sequence of parameters $\langle \overline{b}_\eta : \eta \in {}^{\omega>}2\rangle$ there exist 1-fbti $\langle \overline{a}_\eta : \eta \in {}^{\omega>}2\rangle$ "modeled" (or "based") on the original set of parameters: i.e. such that*

($*$) for any tuple $\overline{\eta}$ from T and finite set Δ of L-formulas there exists a tuple $\overline{\nu}$ such that $\overline{\nu} \sim_1 \overline{\eta}$ such that $\overline{a}_{\overline{\eta}} \equiv_\Delta \overline{b}_{\overline{\nu}}$.

We will now define $\overline{\eta} \sim_{\gamma,n} \overline{\nu}$. This is confusing: what I mean is $\overline{\eta}$ and $\overline{\nu}$ agree on all values from the root up until and including the point γ. This means they have the same restriction to ${}^{\gamma+1}2$. I looked at "${}^{\leq\gamma}2$" and it confused me too many times into thinking it meant ${}^{\leq\gamma+1}2$, so I just wrote down the argument that way.

Definition 2.12 (in [1])**.** *Let $L_{\gamma,n}$ have a signature extending that of L_T by the following:*

(1) a function $f(x,y)$ that takes two elements to their meet

(2) a relation $R(x,y)$ that holds of a pair of elements if and only if x has length less than y

(3) a constant for each element of ${}^{\gamma+1>}2$

(4) a sequence of relations $P(d,k)$ that hold of $(\eta_0,\dots,\eta_{d-1})$ if and only if $|\{l(\eta_i) : i \in d \wedge l(\eta_i) > \gamma+1\}| = k$.

We say that $\overline{\eta} \sim_{\gamma,n} \overline{\nu}$ if they have the same quantifier-free $L_{\gamma,n}$-type and moreover decide positively for $P(d,k)$ for the same $k \leq n$.

Call the set $\{l(\eta_i) : i \in d \wedge l(\eta_i) > \gamma+1\} =: u(\gamma,n)(\overline{\eta})$ (referred to as u_1 in [1]).

The following is a description of when two tuples of length d have the same $L_{\gamma,n}$-type.

First, define what is called $cl_\gamma(\overline{\eta})$ in [1]: $\overline{\eta}' = \langle\langle\rangle, \eta_0, \eta_o \restriction \gamma, \eta_1, \eta_1 \restriction \gamma, \eta_0 \cap \eta_1, \dots\rangle$. Enumerate this tuple as $\langle \eta'_1, \dots, \eta'_K\rangle$ where K is $\binom{d}{2} + 2d + 1$.

$\overline{\nu}'$ is defined similarly.

η,ν have the same $\sim_{\gamma,n}$-type if their entries at l have the following conditions:

(i) $\eta'_l \in {}^{\leq\gamma+1}2 \leftrightarrow \nu'_l \in {}^{\leq\gamma+1}2$ and if both sides of biconditional are true, then $\eta'_l = \nu'_l$

(ii) $n \geq |u(\gamma,n)(\overline{\eta})| = |u(\gamma,n)(\overline{\eta})|$

(iii) if $lg(\eta'_l), lg(\eta'_k) \in u(\gamma, n)(\overline{\eta})$ *then* $lg(\eta'_l) < lg(\eta'_k) \leftrightarrow lg(\nu'_l) < lg(\nu'_k)$

(iv) $\eta'_l \lhd \eta'_k \leftrightarrow \nu'_l \lhd \nu'_k$

(v) $\eta'_l{}^\frown\langle 0\rangle \lhd \eta'_k \leftrightarrow \nu'_l{}^\frown\langle 0\rangle \lhd \nu'_k$

Here is an abbreviation of these five conditions: we'll use the notation that η is exceptional at l if η'_l has length greather than $\gamma+1$ (we might say η has exceptional length at l).

(i) For every l, η and ν agree on whether they are exceptional at l, and if they are not, then they are identical at l.

(ii) η and ν have an identical number of exceptional lengths and n bounds this number.

(iii) Length-of-indexed-element linearly orders the indices of η and ν (not necessarily compatible with the order on the integers in d), and the identity map on d is an order-isomorphism with respect to this new order.

(iv) The partial tree order is preserved. (as in $\sim_1$)

(v) Directionality is preserved. (as in $\sim_1$)

Definition 2.12 (from [1]). *Say the sequence* $(b_\eta : \eta \in {}^\omega 2)$ *is*

(1) (γ, n)*-fbti if for all* d*,*

$$\eta_0, \ldots, \eta_{d-1} \sim_{\gamma,n+1} \nu_0, \ldots, \nu_{d-1} \Rightarrow \overline{b}_{\overline{\eta}} \equiv_\Delta \overline{b}_{\overline{\nu}}$$

(2) (ω, n)*-fbti if for all* γ *and* d*,*

$$\eta_0, \ldots, \eta_{d-1} \sim_{\gamma,n} \nu_0, \ldots, \nu_{d-1} \Rightarrow \overline{b}_{\overline{\eta}} \equiv_\Delta \overline{b}_{\overline{\nu}}$$

(3) 0-fbti if it is (ω, n)*-fbti for all* n

Some Definitions for the Ramsey Theory bit:

[**all** quoted from Todorcevic, [2]]

Definition 2.12. *Given a tree* T *and* $x \in T$, $T[x]$ *is the set of elements in* T *related to* x *under the partial tree-order.* $T(k)$ *is the set of elements in* T *at level* k*.*

Definition 3. *A product of trees* $\prod_{i<d} T_i$ *contains a* somewhere-dense matrix *if there exist* $x_i \in X_i \subset T_i$ *and* $k >$ *max length*(x_i) *such that each* X_i *is* $k - x_i$*-dense in* T_i*:*

i.e. X_i *dominates (under the tree order,* $\leq_{T_i}$*) every member of* $T_i[x_i] \cap T_i(k)$ *– those are the level-k members of the subtree of elements of* T_i *related to* x_i*.*

Theorem 4 (Halpern-Laüchli; as in [2])**.** *For every finite sequence* T_i *(*$i < d$*) of finitely-branching, rooted trees and every finite partition of their product* $\prod_{i<d} T_i$*, one of the pieces of the partition must contain a somewhere-dense matrix.*

Corollary 5 (Finite Halpern-Laüchli; as in [2])**.** *For every* $l \geq 1$ *there is an* n *such that any partition* $c : \prod T_i \restriction n \to l$ *is constant on a somewhere dense matrix* $\prod X_i$ *with* $X_i \subseteq T_i \restriction n$*.*

Definition 6. *For a finitely-branching, rooted tree* T*,* $S \subset T$ *is a* strong subtree witnessed by A *if* $A \subseteq \omega$ *is some infinite set such that*

(1) $S \subset \cup\{T(n) : n \in A\}$ *and* $S \cap T(n) \neq \emptyset$ *for all* $n \in A$

(2) If $m < n$ *are successive in* A *then for every* $s \in S \cap T(m)$ *every immediate successor of* s *in* T *has exactly one extension to* $S \cap T(n)$

Corollary 7 (Strong Subtree Halpern-Laüchli; as stated in [2])**.** *For every finite partition of* $\prod_{i<d} T_i$ *there is a piece of the partition* P *and a sequence* S_i *of strong subtrees of* T_i *witness by the same set* A *such that* $\cup_{n\in\omega} \prod_{i<d} S_i(n) \subset P$

3. Argument

Here is a small summary of argument:

Fix some Δ. Suppose we can obtain some b's based on the a's indexed by ${}^{\omega>}2$ that are 0-fbti (for Δ). Then we obtain 1-fbti b's as follows: we take a skew subtree of ${}^{\omega>}2$ that is still perfect. Then for $\sim_1$-equivalent tuples from this subtree, they are automatically $\sim_{n,\gamma}$-equivalent for some n and γ (γ will be the maximal possible tree of agreement and n will be the exceptional values afterwards, bounded by the length of the tuple.) However, our tree was 0-fbti, so $\sim_{n,\gamma}$-equivalence implies that the associated tuples are Δ-equivalent.

Now we give an outline of the detailed argument, and then we describe each point.

I. We are given a sequence of a's and we are trying to obtain b's that are indiscernible and modeled on the a's.
  A. We get indiscernibility in two parts. We want to show that:
    1. $\overline{\eta} \sim_1 \overline{\nu} \Rightarrow \overline{a}_{\overline{\eta}} \equiv \overline{a}_{\overline{\nu}}$
    2. Note that by compactness it is enough to show that for every $\Delta(x_0, \ldots, x_{d-1}) \subset L$ a finite collection of formulas:
$$\overline{\eta} \sim_1 \overline{\nu} \Rightarrow \overline{a}_{\overline{\eta}} \equiv_\Delta \overline{a}_{\overline{\nu}}$$
    3. In fact we can assume that the length of $\overline{\eta}$ matches that of the length of variables for Δ and also we are taking

tuples of this length from a full binary tree of height some N.

B. This requires that we get a large enough 0-fbti tree based on the original parameters, such that we can take a skew subtree wherein 0-fbt-indiscernibility implies $\sim_1$-indiscernibility.

C. To preserve the modeling condition, we will make sure that our transition from one tree to the next is by taking successive strong subtrees. If we start out with ${}^{\omega}2$, this will cause every subsequent tree to be fully binary, but also based on the previous tree and all the nth branchings will occur on the same level. At every step we are likely to identify our new tree with ${}^{\omega}2$ but will try to be careful about when we are identifying an element as it was called in the supertree, and when we are identifying it as it is now called in the subtree!

D. We want full binary subtrees in the previous so that we get that "big enough subtree of ${}^{\omega}2$ that we mentioned in step IB. We want all the nth branchings in the subtree to occur on the same level of the supertree so that we can proceed in our induction level-by-level and end up with a nice full binary tree (instead of.."we never got this node to branch, ever!")

E. In particular, when we are moving from a (γ, n)-indiscernible to a $(\gamma+1, n)$-indiscernible tree, not only do we want strong subtrees to preserve based-ness, but we want to fix the first γ levels (not including the root) so that our sequence of trees converges into one, nice (ω, n)-indiscernible tree.

II. Here is the structure of the argument for 0-fbtis with the modeling condition. We basically show how to proceed by induction from an (ω, n)-indiscernible tree based on the original parameters to a $(\omega, n+1)$-indiscernible tree.

A. Suppose we have $(\overline{a}_\eta : \eta \in {}^{\omega>}2)$ that is (ω, n)-indiscernible (i.e. it is (γ, n)-indiscernible for all $\gamma < \omega$.) We want to find a strong subtree of ${}^{\omega>}2$ that is $(\omega, n+1)$-indiscernible.

B. In our argument we define:

$$T^{n+1}_{-1} := T(n)$$

where $T(n)$ is the (ω, n)-indiscernible tree.

C. In this case, T^{n+1}_i will be $(i, n+1)$-indiscernible.

D. We show that we can go from T^{n+1}_γ to $T^{n+1}_{\gamma+1}$ such that the $(T^{n+1}_\gamma)_{\gamma<\omega}$ form a fusion sequence.

E. In other words, each is a subtree of the previous and the first γ levels above the common stem of T_γ and $T_{\gamma+1}$ are equal. Thus, the T_γ converge:

$$T^{n+1}_\infty = \bigcap_{\gamma\in\omega} T^{n+1}_\gamma$$

F. We set

$$T(n+1) := T^{n+1}_\infty$$

where $T(n+1)$ is the $(\omega, n+1)$-indiscernible tree.

G. Since $n \leq d$ is bounded by Δ, after finitely many $T(i)$ we are done. Since each is a strong subtree of the previous, the intersection gives us the 0-fbti, isomorphic-to-${}^{\omega}2$ tree, based on the original set of parameters.

III. What does the skew subtree look like exactly.

A. We find subtrees of our 0-fbti tree on ${}^{\omega}2$ that preserves basedness and are "skew of height n"

B. The "skew of height n" tree will have a natural isomorphism with ${}^{\leq N}2$ that preserves basedness.

C. However, in the skew tree, any $\sim_1$-similar pair will be automatically $\sim_{\gamma,n}$-similar for some γ and n.

D. Thus the 0-fbti-ness of the super-tree housing the skew tree becomes 1-fbti-ness in the skew tree which is carried by isomorphism to ${}^{\omega}2$

E. We only need to realize $\sim_1$-indiscernibility for Δ in a height N fully binary tree by compactness.

Explanations:

(II A) Recall that 0-fbti means that the same $\sim_{n,\gamma}$-type for all n, γ implies parameters with same Δ-type, i.e.

$$\eta_0, \ldots, \eta_{d-1} \sim_{\gamma,n+1} \nu_0, \ldots, \nu_{d-1} \Rightarrow \bar{b}_{\bar{\eta}} \equiv_\Delta \bar{b}_{\bar{\nu}}$$

We do this by induction on n. For every n we show true for all γ by induction on γ. Each step has its indexed tree.

case n=0: given $(\bar{a}_\eta : \eta \in {}^{\omega>}2)$ this sequence is automatically $(\omega, 0)$-indiscernible, since this is to say that two identical tuples have the same Δ-type. So we can set $T(0) := {}^{\omega}2$.

case n+1: Suppose we are given $(\bar{a}_\eta : \eta \in T(n))$ which is (ω, n)-indiscernible. We want to get a strong subtree of $T(n)$, $T(n+1)$ which is $(\omega, n+1)$-indiscernible. We do this in stages by induction on γ. Set $T^{n+1}_{-1} := T(n)$ our starting tree. We will obtain trees T^{n+1}_γ for $\gamma \geq 0$ which will be $(\gamma, n+1)$-indiscernible, a strong subtree of $T^{n+1}_{\gamma-1}$ and agreeing with this previous tree on the values at $\{-1, 0, 1, \dots, \gamma - 1\}$ (each η in ${}^{\omega}2$ has value "the root" at -1.) This preserves (ω, γ)-indiscernibility from lower trees to higher trees, $T^{n+1}_{\gamma+k}$.

(II D) Here we argue from $T^{n+1}_{\gamma-1}$ to T^{n+1}_{γ}:

We are given $(\overline{a}_\eta : \eta \in T^{n+1}_{\gamma-1})$ and this sequence is (ω, n)-indiscernible (as well as $(\gamma-1, n+1)$-indiscernible, but this is not needed). In the following, identify $T^{n+1}_{\gamma-1}$ with ${}^{\omega}2$. We are going to find a strong subtree T^{n+1}_{γ} indexing a $(\gamma, n+1)$-indiscernible sequence that agrees with $T^{n+1}_{\gamma-1}$ up until values at γ (i.e., the γ-th level remains fixed from $T^{n+1}_{\gamma-1}$ to T^{n+1}_{γ}; the $\gamma + 1$-level will possibly move up.)

In this new tree we want that:

$$\eta_0, \dots, \eta_{d-1} \sim_{\gamma,n+1} \nu_0, \dots, \nu_{d-1} \Rightarrow \overline{b}_{\overline{\eta}} \equiv_\Delta \overline{b}_{\overline{\nu}}$$

Note that there are finitely many classes $Y = [\overline{\eta}]/\sim_{\gamma,n+1}, Y_1, \dots, Y_p$ (this is by inspection of the language $L_{\gamma,n+1}$ which is bounded by functions in γ and n.

We have in $T^{n+1}_{\gamma-1}$ one copy of ${}^{\omega>}2$ on top of each element of ${}^{\gamma+2}2$, Call these trees $T_1, \dots T_{2^{\gamma+1}}$. We want to color every level tuple from this product of trees. Suppose we have a tuple from level $\mu \geq 0$ above the root (this is level $(\gamma+1)+\mu$ according to $T^{n+1}_{\gamma-1}$),

$$\overline{e} = (e_1, \dots, e_{2^\gamma}) \in ({}^{\mu}2)^{2^\gamma}$$

We send $(e_1, \dots, e_{2^\gamma})$ to the color $(c_1, \dots, c_p)$ where each c_i is a Δ-type. We will describe how we choose the Δ-type and why this map is well-defined in $(*)$

First note that for any $\sim_{\gamma,n+1}$-class Y and $\overline{\eta} = \langle \eta_0, \dots, \eta_{d-1} \rangle$ in Y we have an equivalence on indices from $\{0, \dots, d-1\}$. For $u(\gamma, n+1, \overline{\eta})$ we say that:

$$[i]/\sim_Y := \{j : \eta'_i \restriction \min u(\gamma, n+1, \overline{\eta}) = \eta'_j \restriction \min u(\gamma, n+1, \overline{\eta})\}$$

Claim 8. *This relation is independent of the choice of $\overline{\eta}$.*

Proof. Let $\overline{\nu}$ be also in Y with associated u_2. So $\overline{\nu} \sim_{\gamma,n+1} \overline{\eta}$. Let $\rho = \min u(\gamma, n+1, \overline{\nu})$, $\sigma = \min u(\gamma, n+1, \overline{\eta})$.

We need to show:

$$\eta'_i \upharpoonright \mu = \eta'_j \upharpoonright \mu \leftrightarrow \nu'_i \upharpoonright \rho = \nu'_j \upharpoonright \rho$$

This is symmetric, so we do right-to-left. Suppose $\eta'_i \upharpoonright \sigma \neq \eta'_j \upharpoonright \sigma$. In the first case, they have different restrictions to $\gamma + 1$, and then the $\sim_{\gamma,n+1}$-similarity guarantees that ν has the same property at i, j. In the second case, they have the same restrictions to $\gamma + 1$, then the meet $\eta'_i \wedge \eta'_j$ must be in ${}^{\gamma+1}2$. If it is any higher in the tree, then it has to be in ${}^{\mu}2$ (but it's not) because otherwise, since the meet is itself an η'_k and of length less than η'_i it would have exceptional length less than σ, a contradiction.

Well, $\nu'_i \wedge \nu'_j$ is guaranteed by the similarity to also be in ${}^{\gamma+1}2$, which means that the restrictions of the ν' to ρ at i, j must be different (in fact we know the values at $\gamma + 1$ must already be different, and $\rho \geq \gamma + 2$.) □

Let ν^Y be the set $\{[i] : i \in d\}$. If we let $n_Y = |\nu^Y|$ we can list $\nu^Y = \langle s_1^Y, \dots, s_{n^Y}^Y \rangle$.

(Explanation of $*$) 1. We choose the Δ-type c_i on the basis of (i) a $(\gamma, n+1)$-class Y_i and (ii) the tuple $(e_1, \dots, e_{2^\gamma})$. Consider any tuple $\overline{\eta} = \langle \eta_0, \dots, \eta_{d-1} \rangle$ in Y_i such that $(\mu - 1) + (\gamma + 1) = \min u(\gamma, n + 1)(\overline{\eta})$ and $e_{s_k} = \eta_{s_k} \upharpoonright (\gamma + 1) + \mu$.

Let $c_i = \mathrm{tp}_\Delta(\overline{a}_{\overline{\eta}})$.

2. This choice is well-defined. Take any other $\overline{\nu} \sim_{\gamma,n+1} \overline{\eta}$, i.e. in Y_i. Let $\min u(\gamma, n+1)(\overline{\nu}) = (\gamma+1)+(\rho-1)$ and $e_{s_k} = \nu_{s_k} \upharpoonright (\gamma+1)+\rho$. Then we want to conclude that $\overline{\eta} \sim_{(\gamma+1)+(\rho-1),n} \overline{\nu}$. This will cause both $\overline{\mu}$ and $\overline{\nu}$ to be mapped to the same Δ-type c_i.

We only need to show that the two tuples look the same restricted to $(\gamma + 1) + \rho$, as (iv) and (v) are independent of γ and n, (iii) the relative heights is naturally preserved and (ii) the bound on the exceptional heights follows immediately given that we are making the least exceptional height from before, $(\gamma + 1) + (\rho - 1)$, now non-exceptional. But for (i) it is also clear because all the restrictions to $(\gamma + 1) + \rho$, index-by-index, are the common values e_{s_k}. And we know the indices of $\overline{\eta}, \overline{\nu}$ at which they occur are common, since they depend entirely on the $\sim_{\gamma,n+1}$-class Y_i.

(III): We may assume that $(\overline{a}_\eta : \eta \in {}^{\omega>}2)$ are 0-fbti for Δ and that we are looking for $(\overline{b}_\eta : \eta \in {}^{\omega>}2)$ that are 1-fbti and based on the a's.

We use the construction in [1] to skew the tree to the left. What we actually do is find finite skew subtrees of arbitrary height n. This construction preserves the linear order as well as the tree order, however it is such that you can read all of (i)-(iii) off of these two orders. So in other words we have that $\overline{\eta}$ and $\overline{\nu}$ having the same $\sim_1$-type in ${}^{\leq n}2$ have images in the skew tree of height n with the same $\sim_{0,n*}$-type where $n*$ is big enough to bound the u_i of condition (iii).

Here is the construction:

for each n we have a function

$$h_n : {}^{n\geq}2 \to {}^{\omega>}2$$

(this is going to be our isomorphism between ${}^{n\geq}2$ and our skew subtree of height n.)

and we have a number (it is “+1” in [1], but let's make it “+2” here)*

$$k_n = \max\{\lg(h_n(\eta)) + 2 : \eta \in {}^{n\geq}2\}$$

The idea is that every time the original tree turns right, the skew tree copies the old tree, but when it turns left, the skew tree adds on enough zeroes so that no two forks are on the same level – in fact any fork (meaning the node preceding a fork) is above any fork to its right.

Here is the definition:

$$h_0(\langle\rangle) = \langle\rangle$$

$$h_{n+1}(\langle\rangle) = \langle\rangle$$
$$h_{n+1}(\langle 1\rangle^\frown\nu) = \langle 1\rangle^\frown h_n(\nu)$$
$$h_{n+1}(\langle 0\rangle^\frown\nu) = \langle 0,0,\ldots,0\rangle^\frown h_n(\nu)$$

where there are k_n zeroes in the last expression.

Thus given $\overline{\eta}$, $\overline{\nu}$ both in ${}^{n\geq}2$ and $n* =$ length of $cl(\overline{\eta})$,

$$(*2)\overline{\eta} \sim_1 \overline{\nu} \to h_n(\overline{\eta}) \sim_{0,n*} h_n(\overline{\nu})$$

so that two tuples $\sim_1$-similar in the skew tree have $\sim_{0,n*}$-similar images as considered in the super- 0-fbti tree.

Verification of $(*2)$:

(i) We only need to check for $\gamma = 0$. But if the images of η'_0, η'_1 have empty meet in the skew tree than the originals have empty meet in the old tree. This is because the skewing preserves the $\sim_1$-language.

(ii) By definition of $n*$ on the basis of h_n.

(iii) Draw a few trees and check any pair that are $\sim_1$-equivalent, are they $\sim_{0,n*}$-equivalent? Unfortunately I don't have a better argument here right now.

$*$ Note that with the "+1" in the definition of k_n we have the following problem:

look at the image of the tree ${}^{<2}2$. Look at the pairs $(h_2(00), h_2(01)) =: (a, b)$ and $(h_2(0), h_2(10)) =: (c, d)$. $l(a) < l(b)$ but $l(c) = l(d)$. So we should find that the pairs have different $\sim_1$-types. But they seem to agree on all parts of the language, except the meet of the first pair is not the empty sequence and the meet of the second pair is. However this tree will be copied on the right branch of a bigger tree where this won't be a disagreeing factor.

References

[1] Džamonja, Shelah, *On $\triangleleft^*$-maximality*, Ann. Pure Appl. Logic, 125 (2004), no. 1-3, 119-158.

[2] Todorcevic, *Introduction to Ramsey spaces*, Princeton University Press (to appear).