APPENDIX TO "SOME OBSTRUCTIONS TO POSITIVE SCALAR CURVATURE (psc) ON A NONCOMPACT MANIFOLD"

I. BACKGROUND ON SCALAR CURVATURE
II. CONJECTURE RELATING SCALAR CURV. TO SIMPLICIAL VOLUME
III. SOME RESULTS AND THE GLUING ISSUE
IV. HOW TO PROVE THINGS ABOUT SCALAR CURV.

M^n SMOOTH MFLD. WITH A RIEMANNIAN METRIC g, I.E. $\forall m \in M$, HAVE INNER PRODUCT ON $T_m M$.

SECTIONAL CURVATURE: $\forall m \in M$ AND ANY 2-PLANE $P \subset T_m M$, ASSIGN NUMBER $K(P) \in \mathbb{R}$.

SCALAR CURVATURE $R \in C^\infty(M)$.

$R(m) = n(n-1) \cdot$ (AVERAGE OF $K(P)$ AMONG ALL $P \subset T_m M$).

IF $n=2$, $R = 2 \cdot$ (GAUSSIAN CURVATURE).

OPEN QUESTION: WHICH COMPACT M ADMIT A METRIC WITH $R>0$?

KNOWN IF $\pi_1(M) = \{e\}$, GROMOV-LAWSON + STOLZ

CONJECTURE: IF M IS ASPHERICAL (I.E. $\widetilde{M}$ CONTRACTIBLE) THEN M DOES NOT ADMIT A psc METRIC

GENERALIZED CONJECTURE: SAY $\pi = \pi_1(M)$,

HAVE CLASSIFYING SPACE $B\pi$ AND MAP $\nu : M \to B\pi$, ISOMORPHISM ON π_1.

CONJ: SAY M ORIENTED. IF $\nu_*([M]) \neq 0$ IN $H_n(B\pi;\mathbb{Q})$ THEN M DOES NOT ADMIT A psc METRIC.

ALMOST NONNEGATIVE SCALAR CURVATURE: ALLOW SOME NEGATIVE SCALAR CURV., RELATIVE TO, E.G., DIAMETER OR VOLUME

CONJ. (GROMOV 1986) $\forall n \in \mathbb{Z}^+$, $\exists c_n > 0$ SO IF M IS COMPACT CONNECTED ORIENTED n-DIMENSIONAL RIEM. MFLD, WITH $R \geq -\sigma^2$, THEN $\|M\| \leq c_n \sigma^n \operatorname{VOL}(M)$.

TWO WAYS TO THINK ABOUT THIS:

1. RESCALE SO $\sigma = 1$. THEN $\|M\|$ IS OBSTRUCTION TO VOLUME-COLLAPSING WITH $R \geq -1$.
2. RESCALE SO $\operatorname{VOL}(M) = \underline{1}$. THEN $\|M\|$ IS AN OBSTRUCTION TO ALMOST NONNEGATIVE SCALAR CURVATURE, W.R.T. VOLUME NORMALIZATION

POSITIVE RESULTS:

1. TRUE IF SCALAR CURVATURE IS REPLACED BY RICCI CURVATURE (GROMOV)
2. TRUE IF SCALAR CURV. IS REPLACED BY "MACROSCOPIC SCALAR CURVATURE" (BRAUN-SAUER 2021)

ON OTHER HAND, CONJECTURE OPEN

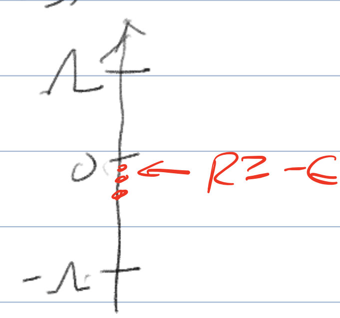
EVEN IF $\sigma = 0$.

PROP. IF M IS COMPACT, SPIN, $R \geq 0$, AND $\pi_1(M)$ SATISFIES STRONG NOVIKOV CONJECTURE (FOR C^*_{max}) THEN $\|M\| = 0$.

PF. RUN RICCI FLOW ON (M,g). EITHER $Ric(M) = 0$, OR IT ACQUIRES $R > 0$.
IF $RIC(M) = 0$, $\pi_1(M)$ IS VIRTUALLY ABELIAN, SO $\|M\| = 0$. IF $R > 0$, SAY $\nu : M \to B\pi$ IS CLASSIFYING MAP.
SNC $\Rightarrow$ $\nu_*(*\hat{A}(M)) = 0$ IN $H_*(B\pi;\mathbb{Q})$
IN PARTICULAR, $\nu_*(*1) = \nu_*[M] = 0$ IN $H_n(B\pi;\mathbb{Q})$. THEN $\|M\| = 0$.

ADDITIONAL GEOMETRIC ASSUMPTIONS:
QUANTITATIVE BOUND ON $\|$SECT. CURV.$\|$

PROP. $\forall\, n \in \mathbb{Z}^+$, $D, \Lambda < \infty$, $\exists\, \epsilon = \epsilon(n, D, \Lambda) > 0$ SO FOLLOWING HOLDS. SAY (M^n, g) IS COMPACT CONNECTED SPIN MFD WITH $\pi_1(M)$ SATISFYING SNC. SUPPOSE

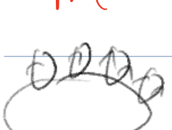


1. $\operatorname{diam}(M,g) \leq D$
2. $|$sect. curv.$| \leq \Lambda$
3. $R \geq -\epsilon$.

THEN $\|M\| = 0$.

PF. SUPPOSE NOT. $\exists$ SEQUENCE (M_i, g_i) WITH $\operatorname{diam}(M_i) \leq D$, $|$sect. curv.$| \leq \Lambda$, $R \geq -\frac{1}{i}$, BUT $\|M_i\| \neq 0$.

M_i

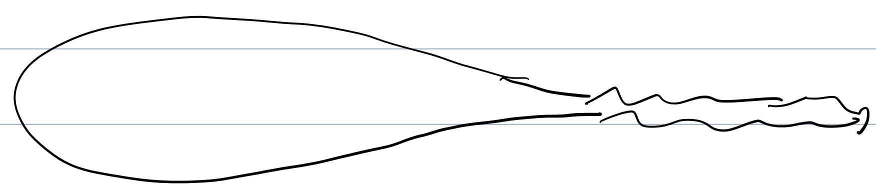


IF $\exists$ SUBSEQUENCE SO $\operatorname{vol}(M_i, g_i) \xrightarrow{i} 0$,

THEN FOR LARGE i, M_i HAS AN AMENABLE COVER OF MULTIPLICITY $\leq n$.
THEN $\|M_i\| = 0$, CONTRADICTION.
HENCE $\exists$ LOWER BOUND $vol(M_i, g_i) \geq V_0 > 0$
THEN CONVERGENT SUBSEQ. $(M_i, g_i) \to (M^n_\infty, g_\infty)$. THEN $R(M_\infty, g_\infty) \geq 0$, SO $\|M_\infty\| = 0$. BUT FOR LARGE i, M_i IS DIFFEOMORPHIC TO M_∞, SO $\|M_i\| = 0$, CONTRADICTION.

RMK: CANNOT REMOVE SECT. CURV. BOUND.
ANY M^n, $n \geq 3$, HAS A SEQUENCE OF RIEM. METRICS SO $\|R(g_i)\| \cdot diam(M_i, g_i)^2 \to 0$. (LOHKAMP 1999).

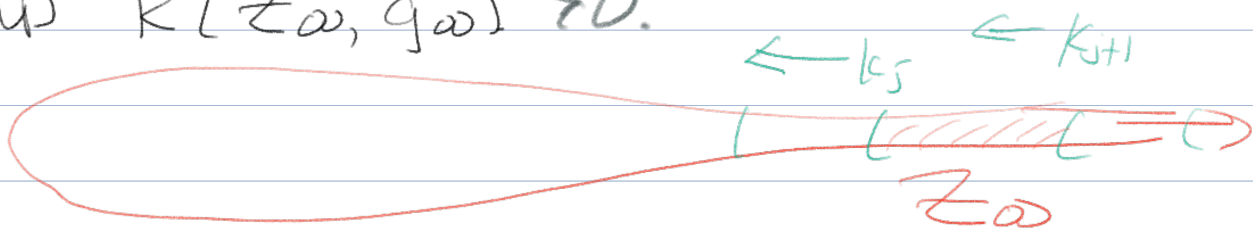
SUPPOSE INSTEAD $vol(M) = 1$, $|sect. curv| \leq \Lambda$
IS THERE SOME $\epsilon(n, \Lambda) > 0$ SO $R \geq -\epsilon \Rightarrow \|M\| = 0$?
SUPPOSE NOT. $\exists$ SEQUENCE (M_i, g_i) WITH $vol(M_i, g_i) = 1$, $|sect. curv| \leq \Lambda$, $R(M_i, g_i) \geq -\frac{1}{i}$, BUT $\|M_i\| \neq 0$.
CAN ASSUME $diam(M_i, g_i) \xrightarrow{i} \infty$.
AFTER PASSING TO A SUBSEQUENCE, FOR LARGE i, GET THICK-THIN DECOMPOSITION OF M_i, I.E. $M_i = M_i^{thick} \cup M_i^{thin}$



M_i

thick

WHERE M_i^{thick} IS DIFFEO. TO A LARGE REGION IN SOME (Z^n_∞, g_∞) AND M_i^{thin} HAS AMENABLE COVER WITH MULTIPLICITY $\leq n$. HERE Z_∞ IS A COMPLETE NONCOMPACT n-MFLD WITH $vol(Z_\infty, g_\infty) < \infty$ AND $R(Z_\infty, g_\infty) > 0$.



HYPOTHESIS: $\exists$ EXHAUSTION $K_1 \subset K_2 \subset \dots$ OF Z_∞ BY COMPACT MFLDS-WITH-BDRY SO $\nu_*[K_{j+1}, K_{j+1} - Int(K_j)]$ VANISHES IN $H_n(B\pi_1(K_{j+1}), B\pi_1(K_{j+1} - Int(K_j)); \mathbb{Q})$.

SAY $\hat{K}_{i,j} \subset M_i$ IS DIFFEO. TO K_j.

TAKE $M_i^{thick} = \hat{K}_{i,j+1}$ AND $M_i^{thin} = M_i - Int(\hat{K}_{i,j})$.

THEN $\| M_i^{thick}, M_i^{thick} \cap M_i^{thin} \| = 0$.

WANT TO SAY $\| M_i \| = 0$..?

SPECIAL FEATURES:

1. $\| M_i^{thick}, M_i^{thick} \cap M_i^{thin} \| = 0$ FOR A REASON: $\nu_*([M_i^{thick}, M_i^{thick} \cap M_i^{thin}]) = 0$ IN $H_n(B\pi_1(M_i^{thick}), B\pi_1(M_i^{thick} \cap M_i^{thin}))$.
2. $M_i^{thick} \cap M_i^{thin}$ HAS AMENABLE COVER WITH MULTIPLICITY $\leq n$.