COMPLEX ALEXANDROV SPACES

SECT. CURV. $\geq K \rightsquigarrow$ TRIANGLE COMPARISON,
ALEXANDROV SPACES, ...

HOLOMORPHIC BISECTIONAL CURVATURE $\geq K \rightsquigarrow$?

1. DEFN. OF HOLO. BISEC. CURV.
2. FACTS ABOUT SECT. CURV.
3. COMPARISON GEOMETRY OF HOLO. BISEC. CURV.
4. LIMIT SPACES
5. SINGULAR SPACES
6. TANGENT CONES

1. SECT. CURV.: M RIEM. MFLD, $p \in M$, $\sigma \subset T_pM$ 2-PLANE. SAY $\sigma = \mathrm{span}(X, Y)$, $|X| = |Y| = 1$, $\langle X, Y \rangle = 0$. THEN $K(\sigma) = R(X, Y, X, Y)$.
HOLO. BISEC. CURV: M KÄHLER MFLD WITH COMPLEX STRUCTURE J. $p \in M$, SAY σ, σ' COMPLEX LINES IN T_pM, I.E. $\sigma = \mathrm{span}(X, JX)$, $\sigma' = \mathrm{span}(Y, JY)$, $|X| = |Y| = 1$. THEN $H(\sigma, \sigma') = R(X, JX, Y, JY)$ HOLO. BISEC. CURV. WRITE $BK \geq K$ IF $H(\sigma, \sigma') \geq K$ FOR ALL p, σ, σ'. BIANCHI $\Rightarrow$
$$R(X, JX, Y, JY) = R(X, Y, X, Y) + R(X, JY, X, JY).$$
(Sect. curv. $\geq$ const.) $\Rightarrow$ (holo. bisec. curv. $\geq$ const.) $\geq$ (Ricci $\geq$ const.)

2. RIEMANNIAN COMPARISON GEOMETRY..

SUPPOSE M COMPLETE RIEM. MFLD, NONNEG. SECT. CURV. GIVEN $p \in M$, SAY $d_p \in C(M)$ IS DISTANCE FROM p. THEN

$$\mathrm{Hess}(d_p^2/2) \le g \quad (\text{OFF CUT-LOCUS OF } p)$$

SAY $\gamma(t)$ IS A UNIT SPEED GEODESIC (AWAY FROM CUT-LOCUS OF p), $0 \le t \le L$.

WRITE $d_p(t) = d_p(\gamma(t))$. THEN

$$\frac{d^2}{dt^2}\left(d_p^2(t)/2\right) \le 1, \text{ I.E.}$$

$$\frac{d^2}{dt^2}\left(d_p^2(t)/2 - t^2/2\right) \le 0, \text{ I.E.}$$

$d_p^2(t) - t^2$ IS CONCAVE IN t. THEN

$$d_p^2(t) \ge \frac{t}{L} d_p^2(L) + \left(1 - \frac{t}{L}\right) d_p^2(0) - t(L-t).$$

TRIANGLE COMPARISON, TRUE GLOBALLY ON M (TOPONOGOV).

3. KÄHLER CASE. M COMPLETE KÄHLER MFLD, ω KÄHLER FORM.

DEFN. $$d_{p,k}^2 = \begin{cases} -\frac{4}{k} \log \cos\left(d_p \sqrt{\tfrac{k}{2}}\right) & \text{IF } k > 0 \\ d_p^2 & \text{IF } k = 0 \\ -\frac{4}{k} \log \cosh\left(d_p \sqrt{-\tfrac{k}{2}}\right) & \text{IF } k < 0 \end{cases}$$

THM. $BK \ge K \Leftrightarrow \sqrt{-1}\,\partial\bar\partial\, d_{p,k}^2/2 \le \omega$, AS CURRENTS ON M, I.E.

$$\partial_\alpha \partial_{\bar\beta} \left(d_{p,k}^2/2\right) \le g_{\alpha\bar\beta}.$$

(CASE $K=0$, $\Rightarrow$ DUE TO CAO-NI.)

FOR SIMPLICITY, ASSUME HEREAFTER $K=0$. INTEGRATE OVER DISKS.

SAY $\Sigma = \text{IMAGE}(\Phi : \overline{\mathbb{R}^2} \to M)$, Φ

CONTINUOUS EMBEDDING, ϕ HOLO. ON B^2.

$BK \geq 0 \Rightarrow \sqrt{-1}\,\partial\bar{\partial}(d_p^2(z)) \leq dA$ ON Σ.

DIRICHLET PROBLEM: IF $\sqrt{-1}\,\partial\bar{\partial}(f(z)) = dA$ ON Σ THEN

$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta + \frac{2}{\pi}\iint \log|z|\,dA.$$

SO $BK \geq 0 \Rightarrow$

$$(*) \qquad d_p^2(0) \geq \frac{1}{2\pi}\int_0^{2\pi} d_p^2(\theta)\,d\theta + \frac{2}{\pi}\iint \log|z|\,dA.$$

(LIKE TRIANGLE COMPARISON)

THM. IF M IS KÄHLER THEN $BK \geq 0$ $\Leftrightarrow$ $(*)$ HOLDS FOR ALL p, Σ.

WHAT IF M IS HERMITIAN?

THM. IF M IS HERMITIAN AND $(*)$ HOLDS THEN M IS KÄHLER.

ANALOGY: IF A FINSLER MFLD. HAS ALEXANDROV CURVATURE BOUNDED BELOW THEN IT'S RIEMANNIAN.

4. LIMIT SPACES

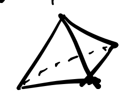
RIEM. CASE: SUPPOSE X IS A NONCOLLAPSED POINTED GH LIMIT OF COMPLETE RIEMANNIAN n-MFLDS WITH NONNEG. SECT. CURV. THEN

1. X IS A TOPOLOGICAL MFLD (PERELMAN)
2. X HAS NONNEG. ALEX. CURV. (BGP)

THM. SUPPOSE $\{(M_i, g_i, p_i)\}_{i=1}^{\infty}$ IS A SEQUENCE OF POINTED COMPLETE KÄHLER

MFLDS WITH $BK \geq 0$ AND
$\mathrm{Vol}(B(p_i,1)) \geq v_0 > 0$ FOR ALL i. THEN
AFTER PASSING TO A SUBSEQUENCE, $\exists$
POINTED GH LIMIT $(X_\infty, d_\infty, p_\infty)$ SO

1. X_∞ IS A COMPLEX MFLD (LEE-TAM)
2. (X_∞, d_∞) SATISFIES $(*)$, WHERE dA IS 2D HAUSDORFF MEASURE ON Σ.

EX. X_∞ CAN BE THE BOUNDARY OF A COMPACT CONVEX BODY IN $\mathbb{R}^3$ 

COR. IF $\lim_{i\to\infty} (M_i^n, g_i, p_i) \overset{PGH}{=} (M^n_\infty, g_\infty, p_\infty)$,
EACH M_i HAS $BK \geq 0$, AND
(M_∞, g_∞) IS SMOOTH THEN (M_∞, g_∞)
IS KÄHLER WITH $BK \geq 0$.

LOCAL POTENTIALS: IN SMOOTH CASE, HAD
$\sqrt{-1}\partial\bar\partial(d_p^2/2) \leq \omega$. LOCALLY, $\omega = \sqrt{-1}\partial\bar\partial\phi$.
THEN $\sqrt{-1}\partial\bar\partial(\phi - d_p^2/2) \geq 0$, I.E.
$\phi - d_p^2/2$ IS PLURISUBHARMONIC (psh).

SINGULAR SPACES

COMPLEX SPACE: LOCALLY MODELLED ON THE ZERO SET OF FINITELY MANY HOLO. FUNCTIONS ON A DOMAIN IN $\mathbb{C}^N$.

KÄHLER SPACE (MOISHEZON): A COMPLEX SPACE WITH AN OPEN COVER $\{U_\alpha\}$ AND

CONTINUOUS psh FUNCTIONS $\{\phi_\alpha\}$ SO $\phi_\alpha - \phi_\beta$ IS PLURIHARMONIC ON $U_\alpha \cap U_\beta$ (~~THEN~~ THEN $\sqrt{-1}\,\partial\bar\partial\phi_\alpha = \sqrt{-1}\,\partial\bar\partial\phi_\beta$ ON $U_\alpha \cap U_\beta$).

DEFN. A METRIC KÄHLER SPACE IS A KÄHLER SPACE AND A METRIC d SO FOR EACH HOLO. DISK $\Sigma \subset X$, $\sqrt{-1}\,\partial\bar\partial\phi_\alpha\big|_{U_\alpha\cap\Sigma}$ IS 2D HAUSDORFF MEASURE ON $U_\alpha \cap \Sigma$ COMING FROM d.

DEFN. A METRIC KÄHLER SPACE HAS "$BK \geq 0$" IF $\forall p \in X$ AND ALL α, $\phi_\alpha - \frac{1}{2} d_p^2$ IS psh ON U_α.

EX. IF $\mathcal{O}$ IS A KÄHLER ORBIFOLD WITH $BK \geq 0$ THEN $|\mathcal{O}|$ HAS "$BK \geq 0$".

CAN DEFINE $\mathbb{C}$-GH-CONVERGENCE.

THM. IF $\lim (X_i, d_i, p_i) = (X_\infty, d_\infty, p_\infty)$ IN $\mathbb{C}$-GH TOPOLOGY, AND EACH (X_i, d_i) HAS "$BK \geq 0$", THEN SO DOES (X_∞, d_∞).

THM. IF $\{(M_i, g_i, p_i)\}$ IS A NONCOLLAPSING SEQUENCE OF POINTED COMPLETE KÄHLER MFLDS. WITH $BK \geq 0$ THEN A SUBSEQUENCE CONVERGES IN $\mathbb{C}$-GH TOPOLOGY.

TANGENT CONES: SAY (X_∞, d_∞) IS A NONCOLLAPSED LIMIT OF RIEM. MFLDS

WITH NONNEG. SECT. CURV. THEN $\forall p \in X_\infty$,

1. T_pX_∞ IS A METRIC CONE (BGP).
2. THE LINK OF THE CONE IS HOMEOMORPHIC TO A SPHERE (KAPOVITCH)
3. THE LINK HAS ALEX. CURV. ≥ 1 (BGP)

NOW SAY (X_∞, d_∞) IS A NONCOLLAPSED LIMIT OF KÄHLER MFLDS. WITH $BK \geq 0$

THM. $\forall p \in X_\infty$, T_pX_∞ IS A KÄHLER CONE BIHOLOMORPHIC TO $\mathbb{C}^n$.

THM. SUPPOSE THAT THE DISTANCE FUNCTION FROM THE VERTEX OF T_pX_∞ IS RADIALLY HOMOGENEOUS IN THE COMPLEX COORDINATES. THEN T_pX_∞ IS AN AFFINE CONE OVER $(\mathbb{CP}^{n-1}, d_{\mathbb{CP}^{n-1}})$, WHERE $(\mathbb{CP}^{n-1}, d_{\mathbb{CP}^{n-1}})$ HAS "$BK \geq 2$".

PROOFS OF STATEMENTS ABOUT LIMIT SPACES USE RICCI FLOW. SINCE WE DIN'T ASSUME AN UPPER CURVATURE BOUND, USE PYRAMID RICCI FLOW.

ANALYZE LIMITING RICCI FLOW AS $T \to 0$.

USE POTENTIAL FLOW ON M_∞ TO

CONSTRUCT LIMITING KÄHLER POTENTIALS ON M_∞, ...