The solution of a Ricci pinching conjecture about 3-manifolds

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Plan of the talk

- Statement of results
- Steps in the proof
- Some details of the proof

Ricci-pinched

Pinching in Riemannian geometry

Given a Riemannian manifold (M, g), to each point $m \in M$ and each 2-plane $\pi \subset T_m M$, we assign a number, the sectional curvature $K(\pi)$.

A spherical space form has a metric g_{round} with constant sectional curvatures $K(\pi) = 1$. If we slightly deform g_{round}, the sectional curvatures will be close to 1.

Question: how much can we deform the sectional curvatures and still know that it's diffeomorphic to a spherical space form?

Note that $\mathbb{C}P^n$, $n \geq 2$, has a Riemannian metric with sectional curvatures covering $\left[\frac{1}{4}, 1\right]$.

Quarter pinching

Quarter pinching conjecture: Let (M, g) be a compact Riemannian manifold. Choose $c \in (\frac{1}{4}, 1]$. Suppose that the sectional curvatures lie between c and 1. Then (M, g) is diffeomorphic to a spherical space form.

Brendle and Schoen proved a version of this that only requires pointwise pinching.

Theorem: (Brendle-Schoen) *Let (M, g) be a compact Riemannian manifold with positive sectional curvature. Choose $c \in (\frac{1}{4}, 1]$. Suppose that for each $m \in M$ and any two 2-planes $\pi_1, \pi_2 \subset T_m M$, we have $K(\pi_1) \geq c\, K(\pi_2)$. Then M is diffeomorphic to a spherical space form.*

(Think of $K(\pi_2)$ as the largest sectional curvature at m.)

Ricci curvature

Given an n-dimensional Riemannian manifold (M, g), the Ricci tensor is a symmetric covariant 2-tensor field Ric with the following property.

Suppose that $\mathbf{v} \in T_m M$ is a unit vector. Then $\mathrm{Ric}(\mathbf{v}, \mathbf{v})$ is $(n-1)$ times the average of the sectional curvatures $K(\pi)$ among all 2-planes containing $\mathbf{v}$.

Suppose that (M, g) has nonnegative Ricci tensor. At each $m \in M$, using the metric to turn the Ricci tensor into a self-adjoint operator on $T_m M$, one can diagonalize it to get eigenvalues

$$0 \leq r_1 \leq \ldots \leq r_n.$$

Ricci pinching

Suppose that (M, g) is an n-dimensional Riemannian manifold with nonnegative Ricci curvature. At each $m \in M$, using the metric to turn the Ricci curvature into a self-adjoint operator on $T_m M$, one can diagonalize it to get eigenvalues

$$0 \leq r_1 \leq \ldots \leq r_n.$$

Definition

Given $c > 0$, the Ricci curvature is c-pinched if for all $m \in M$, we have

$$r_1 \geq c r_n.$$

The conjecture

$$0 \leq r_1 \leq \ldots \leq r_n.$$

Definition

Given $c > 0$, the Ricci curvature is c-pinched if for all $m \in M$, we have

$$r_1 \geq c r_n.$$

The conjecture (apparently due to Hamilton).

Conjecture: *Let (M, g) be a complete Riemannian 3-manifold with nonnegative c-pinched Ricci curvature for some $c > 0$. Then (M, g) is flat or M is compact.*

Now proved from the combined efforts of the speaker, Deruelle-Schulze-Simon and Lee-Topping.

The theorem

Theorem: *Let* (M, g) *be a complete Riemannian* 3*-manifold with nonnegative* c*-pinched Ricci curvature for some* $c > 0$*. Then* (M, g) *is flat or* M *is compact.*

In particular, if (M, g) has positive c-pinched Ricci curvature then M is compact.

Compare with Myers' theorem: If $\mathrm{Ric} \geq (n-1)k^2 g$ then $\mathrm{diam}(M, g) \leq \frac{\pi}{k}$.

So the conjecture is a scale-invariant version of Myers' theorem for 3-manifolds.

Note: in three dimensions Ricci determines Riem, but $\mathrm{Ric} \geq 0$ does not imply $K \geq 0$.

Motivation

A motivation for the conjecture:

Theorem: (Hamilton 1994) *Let M^n be a smooth strictly convex complete hypersurface bounding a region in $\mathbb{R}^{n+1}$. Suppose that its second fundamental form is c-pinched. Then M is compact.*

Hamilton's proof is short and uses the quasiconformality of the Gauss map.

A mean curvature flow proof (assuming bounded second fundamental form) was given by Bourni, Langford and Lynch (2023).

Is there an elementary proof of the Ricci pinching conjecture? The present proof uses Ricci flow.

History of results

Let (M, g) be a complete Riemannian 3-manifold with nonnegative c-pinched Ricci curvature for some $c > 0$. Then (M, g) is flat or M is compact.

Chen-Zhu (2000): True if (M, g) has bounded curvature and nonnegative sectional curvature.

J.L. (2023): True if (M, g) has bounded curvature and sectional curvatures bounded below by $-\frac{C}{r^2}$.

Deruelle-Schulze-Simon: True if (M, g) has bounded curvature.

Lee-Topping: True in general.

Huisken-Koerber: Alternative to DSS step.

Why did it take so long? With nonnegative sectional curvature there are special tools that don't generalize. Need new methods.

Ricci-pinched

Intuition

Let (M, g) be a complete Riemannian 3-manifold with nonnegative c-pinched Ricci curvature for some $c > 0$. Then (M, g) is flat or M is compact.

Suppose that (M, g) isn't flat and is strictly conical outside of a compact set K.

On $M - K$, we have $\mathrm{Ric}(\partial_r, \partial_r) = 0$, so $\mathrm{Ric} = 0$ on $M - K$, so g is flat on $M - K$.

The link L of the cone has constant curvature 1 and must be connected (Cheeger-Gromoll).

So it is S^2 or RP^2, but RP^2 doesn't bound, so $L = S^2$. From Bishop-Gromov, $(M, g) = \mathbb{R}^3$, contradiction.

Why Ricci flow?

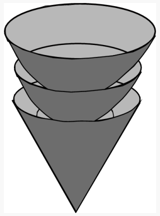
Why does Ricci flow help?

Let's say that the c-Ricci pinched 3-manifold (M, g) is noncompact and nonflat. *Suppose* that we can run a Ricci flow for all time, with initial metric (M, g).

Since Ricci flow is smoothing, we can *hope* that for large time the small scale structure is ironed out, so that we are close to the strictly conical case.

One main issue is to show that the tangent cone at infinity is three dimensional.

Why Ricci flow?



We can even *hope* that a large time limit is asymptotically *self similar*, i.e. is an expanding soliton.

Here an *expanding soliton* is a special type of Ricci flow solution that evolves by expansion and diffeomorphisms.

Enough is known about such a self similar solution to show that under the c-Ricci pinching assumption, it must be flat $\mathbb{R}^3$.

Then we can *try* to say that the initial metric (M, g) has tangent cone at infinity given by $\mathbb{R}^3$ and hence is flat, a contradiction.

The actual proof proceeds somewhat differently.

Ricci flow I

Using basic Ricci flow results, when the curvature is bounded, we can reduce to the case of *positive* Ricci curvature. The Ricci flow will preserve the positivity and c-pinching of the Ricci curvature (Hamilton 1982).

Theorem: (J.L.) *Let* (M, g_0) *be a complete noncompact Riemannian* 3*-manifold having bounded curvature and positive* c*-pinched Ricci curvature.*

The ensuing Ricci flow solution $(M, g(\cdot))$ *exists for all* $t \geq 0$ *and satisfies*

$$\| \operatorname{Rm}(g(t)) \|_\infty \leq \frac{\text{const.}}{t}.$$

Ricci flow II

Theorem: (J.L.) *Let* $(M, g(\cdot))$ *be a Ricci flow on a noncompact Riemannian* 3*-manifold that exists for all* $t \geq 0$, *with complete time slices.*

Suppose that $\mathrm{Ric} > 0$ *and*

$$\| \mathrm{Rm}(g(t)) \|_\infty \leq \frac{\text{const.}}{t}.$$

Then $(M, g(\cdot))$ *is noncollapsing for large time. That is,*

$$\mathrm{vol}(B_{g(t)}(m_0, \sqrt{t})) \geq \text{const.}\, t^{\frac{3}{2}}.$$

This result does not need c-pinching. Examples of such flows come from asymptotically conical expanding Ricci solitons, which exist in abundance (Deruelle).

Ricci flow III

Corollary: *Let* (M, g) *be a complete noncompact Riemannian 3-manifold having bounded curvature and positive c-pinched Ricci curvature.*

Then (M, g) *has cubic volume growth, i.e.*

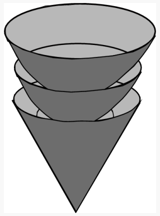
$$\liminf_{r \to \infty} r^{-3} \operatorname{vol}(B(m_0, r)) > 0.$$

Also, we can take blowdown limits of the ensuing Ricci flow.

Proof when $K \geq 0$

Suppose that (M, g) has nonnegative sectional curvature and is c-Ricci pinched. For $s \geq 1$, put $g_s(u) = s^{-1} g(su)$. Let $g_\infty(u) = \lim_{j \to \infty} g_{s_j}(u)$ be a pointed blowdown limit.

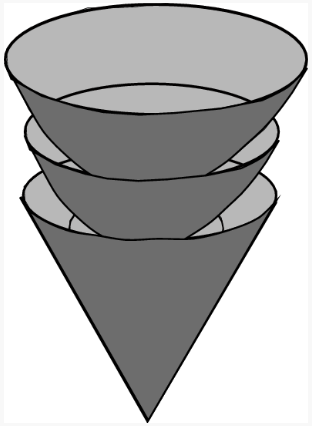
It will be a Ricci flow solution coming out of a cone, namely the tangent cone at infinity $T_\infty M = \lim_{j \to \infty}(M, m_0, s_j^{-\frac{1}{2}} d)$.



By Simon-Schulze, $g_\infty(\cdot)$ is an expanding gradient soliton.

Lemma: *A three dimensional expanding gradient soliton with nonnegative c-pinched Ricci curvature must be flat $\mathbb{R}^3$.*

Proof when $K \geq 0$



Now the tangent cone at infinity of $(M_\infty, g_\infty(u))$ is also equal to $T_\infty M$, so the latter must be $\mathbb{R}^3$.

Theorem: (Colding) *If a complete Riemannian n-manifold* (M, g) *has* $\mathrm{Ric} \geq 0$*, and a tangent cone at infinity isometric to* $\mathbb{R}^n$*, then* (M, g) *is isometric to* $\mathbb{R}^n$*.*

Thus (M, g) is flat, which contradicts our assumption that $\mathrm{Ric} > 0$.

Possibilities and Questions I

How to remove the assumption that $K \geq 0$?

If we could show that the blowdown Ricci flow $g_\infty(u)$ is an expanding gradient soliton then we would be done.

Question: Let $(M, g(\cdot))$ be a Ricci flow on a noncompact Riemannian manifold that exists for all $t \geq 0$, with complete time slices. Suppose that

- $\mathrm{Ric} > 0$,
- $\| \mathrm{Rm}(g(t))\|_\infty \leq \frac{\text{const.}}{t}$, and
- $\mathrm{vol}(B_{g(t)}(m_0, \sqrt{t})) \geq \text{const.}\, t^{\frac{n}{2}}$.

Is a blowdown limit necessarily an expanding soliton?

May want to restrict to three dimensions. In higher dimensions, there are examples of different expanding solitons coming out of the same cone.

Possibilities and Questions II

We know that the tangent cone at infinity $T_\infty M$ of the initial metric is three dimensional and has nonnegative Ricci curvature in a generalized sense.

Then its link L has Ricci curvature bounded below by 1, in the generalized sense (Ketterer).

Since L is a surface, this means that L has Alexandrov curvature bounded below by 1 (Lytchak-Stadler). Hence $T_\infty M$ has nonnegative Alexandrov curvature.

Nonnegative sectional curvature is preserved under 3D Ricci flow starting from a *smooth* Riemannian manifold.

Question: Does a 3D Ricci flow solution evolving out of a nonnegatively curved Alexandrov space have nonnegative sectional curvature?

If so then we can apply the $K \geq 0$ result and we are done.

Possibilities and Questions III

We know that the tangent cone at infinity $T_\infty M$ of the initial metric is three dimensional and has nonnegative Ricci curvature in a generalized sense. It may not be smooth.

If there were a measurable *Ricci tensor* on $T_\infty M$, that behaves well under GH limits, then we could conclude that the cone $T_\infty M$ has c-pinched Ricci curvature and hence must be flat.

Question: Is there a measurable Ricci tensor for noncollapsed Ricci limit spaces, so that Gromov-Hausdorff convergence of spaces implies weak convergence of the Ricci tensors?

There is such a notion when Ricci curvature is replaced by *curvature operator* (Lebedeva-Petrunin). Using this, one can prove Hamilton's conjecture when $K(m) \geq -\frac{\text{const.}}{d(m,m_0)^2}$ (J.L.).

Deruelle-Schulze-Simon I

We have the unknown Ricci flow $g_\infty(u)$ coming out of a metric cone $T_\infty M = \mathrm{cone}(L)$ with nonnegative Alexandrov curvature.

The link L, a surface with Alexandrov curvature bounded below by 1, is a GH limit of smooth surfaces Σ_i with sectional curvature bounded below by 1.

From Deruelle and Schulze-Simon, for each i, there is an expanding soliton coming out of $\mathrm{cone}(\Sigma_i)$.

We can then take a limit of these, to obtain an expanding soliton solution $g_{exp}(u)$ coming out of $\mathrm{cone}(L)$.

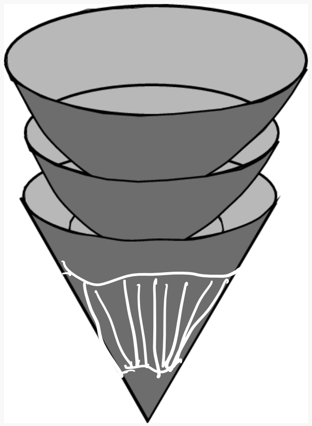
Deruelle-Schulze-Simon II

We now have *two* Ricci flow solutions coming out of the metric cone $T_\infty M$: the blowdown solution $g_\infty(u)$ and the expanding soliton solution $g_{exp}(u)$. If we knew that they're the same, we'd be done.

Theorem: (DSS) *Except at its vertex, the tangent cone at infinity $T_\infty M$ has only regular points. That is, for each $p \in T_\infty M - \{vertex\}$, the tangent cone at p is $\mathbb{R}^3$.*

This comes from the c-Ricci pinching of the flow and a local splitting result of Hochard.

Deruelle-Schulze-Simon III



Localizing in a region away from the vertex, DSS show that since the two Ricci flow solutions have the same initial condition, they must be close for short time.

Theorem: (DSS) *The solutions $\widetilde{g}_\infty(u)$ and $\widetilde{g}_{exp}(u)$ of the Ricci-de Turck equation for the region satisfy*

$$|\widetilde{g}_\infty(u) - \widetilde{g}_{exp}(u)| \leq e^{-\frac{\text{const.}}{u}}.$$

This is a special case of a result that holds in any dimension.

Deruelle-Schulze-Simon IV

Since $g_\infty(u)$ is Ricci c-pinched, it follows that $g_{exp}(u)$ must be almost Ricci c-pinched in the following sense.

Put $g_{sol} = g_{exp}(1)$. It satisfies the expanding gradient soliton equation

$$\mathrm{Ric}_{sol} + \mathrm{Hess}(f) + \frac{1}{2} g_{sol} = 0,$$

where f is the *soliton potential*. The potential goes like $f \sim -\frac{r^2}{4}$.

Theorem: (DSS) *The soliton metric is almost Ricci pinched in the sense that*

$$\mathrm{Ric}_{sol} \geq \mathrm{const.}\, R_{sol} - e^{\mathrm{const.}\, f} g_{sol}.$$

This uses the closeness between $g_\infty(u)$ and $g_{exp}(u)$, along with the self-similar nature of $g_{exp}(u)$.

Deruelle-Schulze-Simon IV

The expanding soliton satisfies

$$\mathrm{Ric}_{sol} \geq \mathrm{const.}\, R_{sol} - e^{\mathrm{const.}\, f} g_{sol}. \tag{1}$$

Proposition: (DSS) *A three dimensional expanding gradient soliton that satisfies (1) must be flat* $\mathbb{R}^3$*.*

Thus $T_\infty M$ is flat $\mathbb{R}^3$ and we conclude as before that $(M, g) = \mathbb{R}^3$. However, we are assuming that (M, g) has positive Ricci curvature. Contradiction.

This proves the c-Ricci pinching conjecture when (M, g) has bounded curvature.

Lee-Topping I

Theorem: (Lee-Topping) *Let (M, g) be a complete noncompact Riemannian 3-manifold with nonnegative c-pinched Ricci curvature for some $c > 0$. Then there is a Ricci flow solution $g(t)$, defined for $t \geq 0$, with $g(0) = g$ so that*

1. *The time slices are complete with nonnegative c-pinched Ricci curvature, and*
2. $|\operatorname{Rm}| \leq \frac{\text{const.}}{t}$.

Note that (M, g) may have unbounded curvature.

Corollary: *Under the hypotheses of the theorem, (M, g) is flat.*

Proof: By DSS, for each $t > 0$, the time-t slice $(M, g(t))$ is flat. Hence (M, g) is flat.

Lee-Topping II

The localized result:

Theorem: (Lee-Topping) *Given $\epsilon \in [0, \frac{1}{12}]$, there are constants $T(\epsilon), a(\epsilon) > 0$ with the following property. Let (M, g) be a complete noncompact Riemannian 3-manifold with $\mathrm{Ric} \geq \epsilon R g \geq 0$. Then for any $m \in M$, there is a Ricci flow solution $g(t)$ on $B_g(m, 1) \times [0, T]$, with $g(0) = g$, so that*

1. $|\mathrm{Rm}| \leq \frac{a}{t}$, *and*
2. $\mathrm{Ric} \geq \epsilon R g - g$.

To prove the global result you start with (M, g), rescale down to $(M, L^{-2} g)$, apply the local result, and parabolically rescale back up. This gives the estimates on $B_g(m, L) \times [0, L^2 T]$, with (2) becoming $\mathrm{Ric} \geq \epsilon R g - L^{-2} g$. Then you can take a convergent sequence of Ricci flow solutions with respect to a sequence $L_j \to \infty$.

Huisken-Koerber I

Theorem: (Huisken-Koerber) *Let* (M, g) *be a complete connected noncompact Riemannian* 3*-manifold with nonnegative* c*-pinched Ricci curvature. If* (M, g) *has cubic volume growth then it is isometric to* $\mathbb{R}^3$*.*

How to use this to prove the conjecture: Suppose that (M, g) is noncompact with nonnegative c-pinched Ricci curvature.

1. Using Lee-Topping, it's enough to assume that (M, g) has bounded curvature and show that it's flat.
2. If it's not flat, we can assume that it has positive Ricci curvature and hence cubic volume growth (J.L.)
3. From Huisken-Koerber, it's isometric to $\mathbb{R}^3$, contradiction.

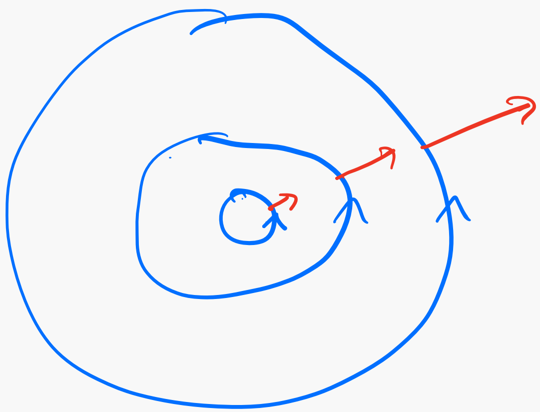
Huisken-Koerber II

Inverse mean curvature flow: a flow of a hypersurface Σ in a Riemannian manifold.

$$\frac{dx}{dt} = \frac{\nu}{H},$$

where

- x is a point on Σ,
- ν is an outward pointing unit normal, and
- H is the mean curvature of Σ at x.



Example: If $M = \mathbb{R}^n$ and the initial hypersurface Σ_0 is a sphere of radius r_0 then Σ_t is a sphere of radius $r_0 e^{\frac{t}{n-1}}$.

Huisken-Koerber III

$$\frac{dx}{dt} = \frac{\nu}{H} \tag{2}$$

Theorem (Huisken-Ilmanen 2001, Moser 2007, Mari-Rigoli-Setti 2022): *Suppose that*

- *M is a complete noncompact Riemannian manifold,*
- *M has nonnegative Ricci curvature and Euclidean volume growth, and*
- *Σ_0 is a compact connected hypersurface.*

Then there is a weak solution $\{\Sigma_t\}_{t\geq 0}$ to (2), starting from Σ_0.

The Σ_t's are compact and connected. If $\dim(M) = 3$ then

$$\frac{d}{dt}\int_{\Sigma_t} H^2 dA = -2\int_{\Sigma_t}\left(\mathrm{Ric}(\nu,\nu) + |A_0|^2\right) dA,$$

where A_0 is the traceless second fundamental form.

Huisken-Koerber IV

Suppose that $\dim(M) = 3$, M has nonnegative Ricci curvature and $\mathrm{Ric} \geq \epsilon R g$.
From Gauss-Bonnet, if genus$(\Sigma) > 0$ then

$$2 \int_\Sigma \left(\mathrm{Ric}(\nu, \nu) + |A_0|^2 \right) dA \geq \int_\Sigma H^2 dA,$$

and if genus$(\Sigma) = 0$ then

$$2 \int_\Sigma \mathrm{Ric}(\nu, \nu) dA \geq \epsilon \left(16\pi - \int_\Sigma H^2 dA \right).$$

Proposition (Mondino 2010): *Suppose that $R(p) > 0$. For small r, the r-sphere $S_r(p)$ has $\int_{S_r(p)} H^2 dA < 16\pi$.*

Huisken-Koerber V

Theorem: (Huisken-Koerber) *Let (M,g) be a complete connected noncompact Riemannian 3-manifold with nonnegative c-pinched Ricci curvature. If (M,g) has cubic volume growth then it is isometric to $\mathbb{R}^3$.*

Proof: Suppose that M is not flat. Choose p with $R(p) > 0$. Choose a small r so that $\int_{S_r(p)} H^2 dA < 16\pi$. Run the IMCF starting from $S_r(p)$. From the previous geometric inequalities, one deduces that $\lim_{t\to\infty} \int_{\Sigma_t} H^2 dA = 0$.

Theorem: (Agostiniani-Fogagnonlo-Mazzieri 2020, X. Wang) *Since M has nonnegative Ricci curvature and cubic volume growth, there is a (sharp) positive lower bound on $\int_\Sigma H^2 dA$ among all compact hypersurfaces Σ.*

This gives a contradiction. Hence M must be flat. Since it has cubic volume growth, it is isometric to $\mathbb{R}^3$.

What about higher dimensions?

In dimension greater than two, there's a result about pinching of the *curvature operator* $\mathrm{Riem} : \Lambda^2 M \to \Lambda^2 M$.

Theorem: (Ni-Wu 2007) *Let* (M^n, g) *be a complete Riemannian manifold with* $n \geq 3$*. Suppose that* $\mathrm{Riem} \geq \epsilon R \,\mathrm{Id} \geq 0$ *for some* $\epsilon > 0$*. Then* M *is compact or flat.*

This assumes $\mathrm{Riem} \geq 0$. What about just $\mathrm{Ric} \geq 0$? One could consider the pinching condition $\mathrm{Ric} \geq \epsilon R g \geq 0$.

From the viewpoint of Ricci flow, it's more natural to look at $\lambda_1 + \lambda_2$, the sum of the lowest two eigenvalues of Riem.

In three dimensions, $\lambda_1 + \lambda_2 \geq 0$ is equivalent to $\mathrm{Ric} \geq 0$. In any dimension, the condition $\lambda_1 + \lambda_2 \geq 0$ is preserved by Ricci flow.

In any dimension, one could consider the pinching condition $\lambda_1 + \lambda_2 \geq \epsilon R \geq 0$.

Ricci-pinched

Ricci flow I

Theorem: *Let (M, g_0) be a complete noncompact Riemannian 3-manifold having bounded curvature and c-pinched positive Ricci curvature.*

The ensuing Ricci flow solution $(M, g(\cdot))$ exists for all $t \geq 0$ and satisfies

$$\| \operatorname{Rm}(g(t)) \|_\infty \leq \frac{\text{const.}}{t}.$$

Put $\sigma = \left(\frac{c}{2+c}\right)^2 \in (0, \frac{1}{9}]$ and

$$f = R^{\sigma-2} \left| \operatorname{Ric} - \frac{1}{3} R g \right|^2.$$

One finds that

$$\left(\frac{\partial}{\partial t} - \triangle \right) f^{\frac{1}{\sigma}} \leq - \frac{2}{3} f^{\frac{2}{\sigma}}.$$

Ricci flow I

From the weak maximum principle,

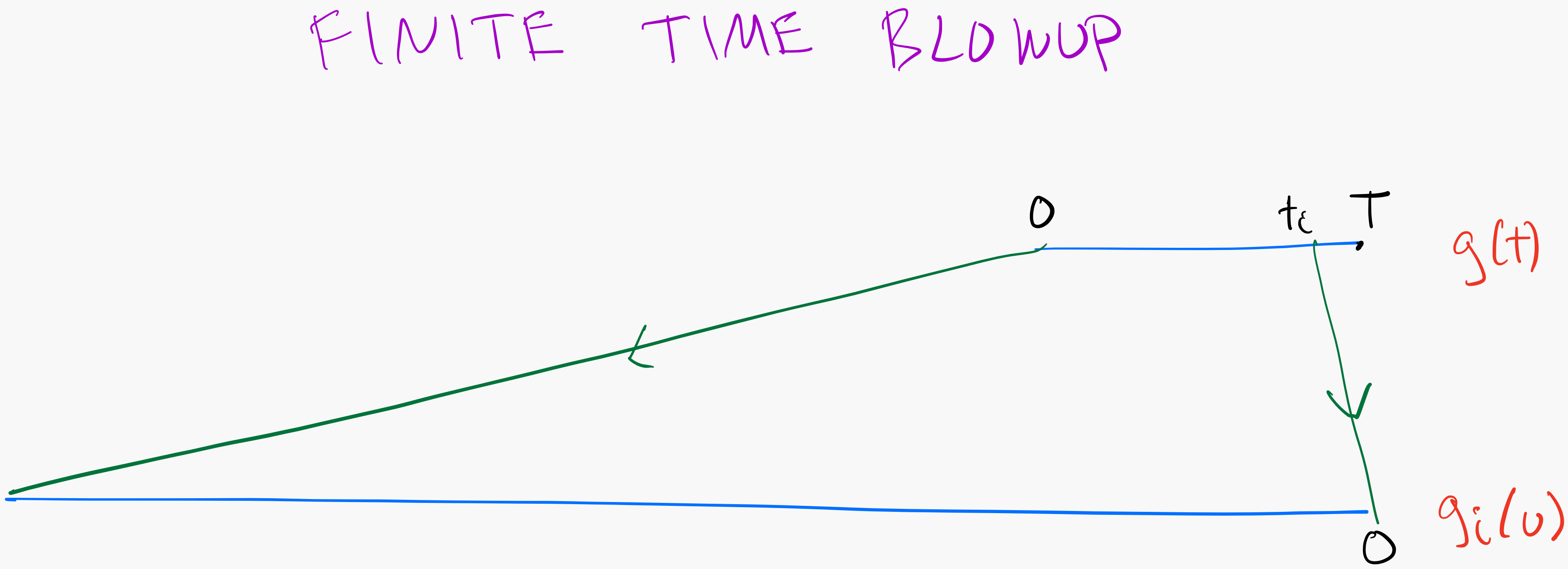
$$\sup_{m \in M} f^{\frac{1}{\sigma}}(m,t) \leq \frac{3}{2t},$$

so

$$R^{-2} \left| \mathrm{Ric} - \frac{1}{3} R g \right|^2 \leq \left(\frac{3}{2tR} \right)^{\sigma}. \tag{3}$$

Suppose that there is a singularity at time $T < \infty$. There is a sequence $\{t_i\}_{i=1}^{\infty}$ of times increasing to T, and points $\{m_i\}_{i=1}^{\infty}$ in M so that $\lim_{i \to \infty} |\mathrm{Rm}(x_i, t_i)| = \infty$ and $|\mathrm{Rm}(m_i, t_i)| \geq \frac{1}{2} \sup_{(m,t) \in M \times [0,t_i]} |\mathrm{Rm}(m,t)|$.

Ricci flow I



Put $Q_i = |\operatorname{Rm}(m_i, t_i)|$ and $g_i(x, u) = Q_i g(x, t_i + Q_i^{-1} u)$. Then g_i is a Ricci flow solution with curvature norm equal to one at $(m_i, 0\ )$, and curvature norm uniformly bounded above by two for $u \in [-Q_i t_i, 0]$.

Ricci flow I

Suppose first that for some $i_0 > 0$ and all i, we have $Q_i \operatorname{inj}_{g(t_i)}(m_i)^2 \geq i_0$. After passing to a subsequence, there is a pointed Cheeger-Hamilton limit

$$\lim_{i\to\infty}(M, g_i(\cdot), m_i) = (M_\infty, g_\infty(\cdot), m_\infty),$$

where $g_\infty(u)$ is defined for $u \in (-\infty, 0]$.

The property of having nonnegative Ricci curvature passes to the limit. By construction, g_∞ has curvature norm one at $(m_\infty, 0)$. Hence g_∞ has positive scalar curvature at $(m_\infty, 0)$. By the strong maximum principle, it follows that g_∞ has positive scalar curvature everywhere.

Ricci flow I

Given $m' \in M_\infty$, the point $(m', 0)$ is the limit of a sequence of points $\{(m'_i, 0)\}_{i=1}^\infty$ with $\lim_{i\to\infty} R_{g_i}(m'_i, 0) = R_{g_\infty}(m', 0) > 0$. As $\lim_{i\to\infty} Q_i = \infty$, after undoing the rescaling it follows that $\lim_{i\to\infty} R_g(m'_i, t_i) = \infty$. As $\lim_{i\to\infty} t_i = T$, we also have $\lim_{i\to\infty} t_i R_g(m'_i, t_i) = \infty$.

Equation (3) implies that the metric $g_\infty(0)$ satisfies $\mathrm{Ric} - \frac{1}{3} R g_\infty(0) = 0$. As $g_\infty(0)$ has positive scalar curvature at $(m_\infty, 0)$, it follows that M_∞ is a spherical space form. Then M is compact, which is a contradiction.

Ricci flow I

Even if there is no uniform lower bound on $Q_i \, \mathrm{inj}_{g(t_i)}(m_i)^2$, after passing to a subsequence we can take a limit to get a Ricci flow on an étale groupoid.

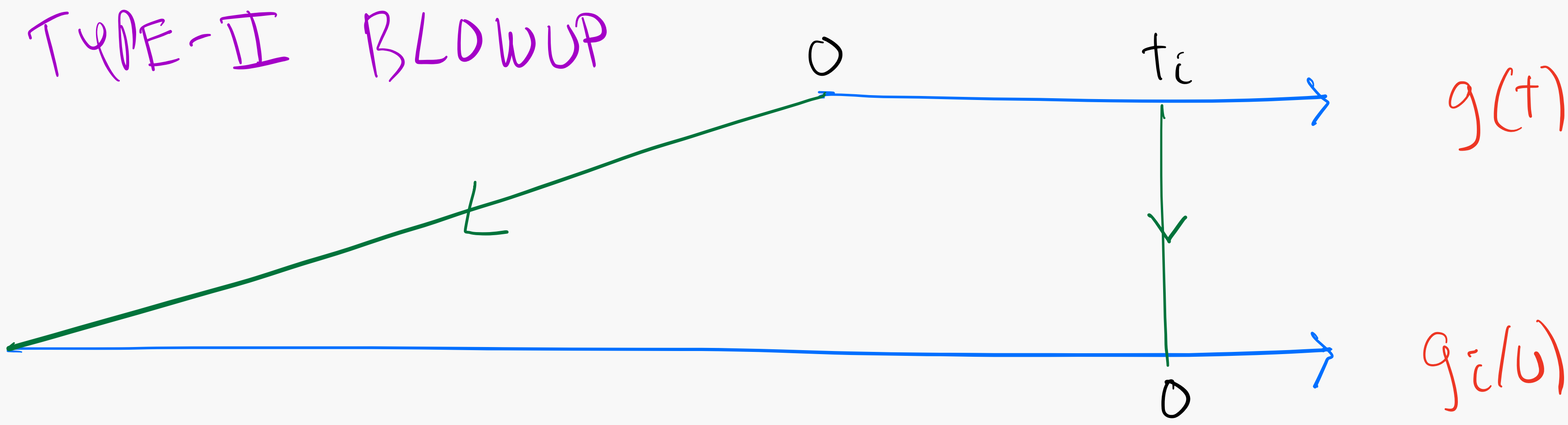
By the same argument, the metric $g_\infty(0)$ on the unit space of the groupoid has constant positive sectional curvature. Then by a Bonnet-Myers argument, the orbit space of the groupoid is compact. It follows that M is compact, which is a contradiction.

We claim now that there is some $C < \infty$ so that for all $t > 0$, we have $\| \operatorname{Rm}(g(t))\|_\infty \leq \frac{C}{t}$.

Suppose not. After doing a type-II point picking, there are points (m_i, t_i) so that $\lim_{i\to\infty} t_i | \operatorname{Rm}(m_i, t_i)| = \infty$ and $| \operatorname{Rm} | \leq 2| \operatorname{Rm}(m_i, t_i)|$ on $M \times [a_i, b_i]$, with $\lim_{i\to\infty} | \operatorname{Rm}(m_i, t_i)|(t_i - a_i) = \lim_{i\to\infty} | \operatorname{Rm}(m_i, t_i)|(b_i - t_i) = \infty$.

Put $Q_i = | \operatorname{Rm}(m_i, t_i)|$ and $g_i(x, u) = Q_i g(x, t_i + Q_i^{-1} u)$.

Ricci flow I



Suppose first that for some $i_0 > 0$ and all i, we have $Q_i \, \mathrm{inj}_{g(t_i)}(m_i)^2 \geq i_0$. After passing to a subsequence, we get a limiting Ricci flow solution $\lim_{i \to \infty} (M, g_i(\cdot), m_i) = (M_\infty, g_\infty(\cdot), m_\infty)$ defined for times $u \in \mathbb{R}$. Here M_∞ is a 3-manifold and $|\operatorname{Rm}(m_\infty, 0)| = 1$.

As before, for each $m' \in M_\infty$, the point $(m', 0)$ is the limit of a sequence of points $(m'_i, 0)$ with $\lim_{i \to \infty} t_i R_g(m'_i, t_i) = \infty$, where the latter statement now comes from the type-II rescaling.

Ricci flow I

From (3), we get $\operatorname{Ric} - \frac{1}{3} R g_\infty = 0$. Then (M_∞, g_∞) has constant positive curvature time slices, which implies that M_∞ is compact. Then M is also compact, which is a contradiction.

If $\liminf_{i \to \infty} Q_i \operatorname{inj}_{g(t_i)}(m_i)^2 = 0$, we can still take a limit in the sense of étale groupoids. As before, we conclude that M is compact, which is a contradiction.

Distance distortion estimates

Let $d_t : M \times M \to \mathbb{R}$ be the distance function on M with respect to the Riemannian metric $g(t)$. In particular, d_0 ~~be~~ is the distance function with respect to g_0.

Lemma: *There is some $C' < \infty$ so that whenever $0 \leq t_1 \leq t_2 < \infty$, we have*

$$d_{t_1} - C' \left(\sqrt{t_2} - \sqrt{t_1} \right) \leq d_{t_2} \leq d_{t_1}. \tag{4}$$

Fix $m_0 \in M$. Given $s > 0$, put $g_s(u) = s^{-1} g(su)$. Its distance function at time u is $\widehat{d}_{s,u} = s^{-\frac{1}{2}} d_{su}$. From (4), we have

$$\frac{1}{\sqrt{s}} d_0 - C' \sqrt{u} \leq \widehat{d}_{s,u} \leq \frac{1}{\sqrt{s}} d_0. \tag{5}$$

Also, $\| \operatorname{Rm}(g_s(u)) \| \leq \frac{C}{u}$.

Ricci flow II

Theorem: *Let* $(M, g(\cdot))$ *be a Ricci flow on a noncompact Riemannian* 3*-manifold that exists for all* $t \geq 0$*, with complete time slices.*

Suppose that $\mathrm{Ric} > 0$ *and*

$$\| \mathrm{Rm}(g(t))\|_\infty \leq \frac{\text{const.}}{t}.$$

Then $(M, g(\cdot))$ *is noncollapsing for large time. That is,*

$$\mathrm{vol}(B_{g(t)}(m_0, \sqrt{t})) \geq \text{const.}\, t^{\frac{3}{2}}.$$

Ricci flow II

Given a sequence $\{s_i\}_{i=1}^{\infty}$ tending to infinity, after passing to a subsequence we can assume that there is a pointed Gromov-Hausdorff limit of the time-one slices of rescaled Ricci flows: $\lim_{i\to\infty}(M, \widehat{d}_{s_i,1}, m_0) = (X_\infty, d_{X_\infty}, x_\infty)$.

Since M is noncompact, X_∞ is also noncompact. In particular, $\dim(X_\infty) > 0$.

We want to show that there is some sequence with a three dimensional pointed Gromov-Hausdorff limit. If not then we will eventually get a contradiction to the fact that M is diffeomorphic to $\mathbb{R}^3$ (Schoen-Yau).

Ricci flow II

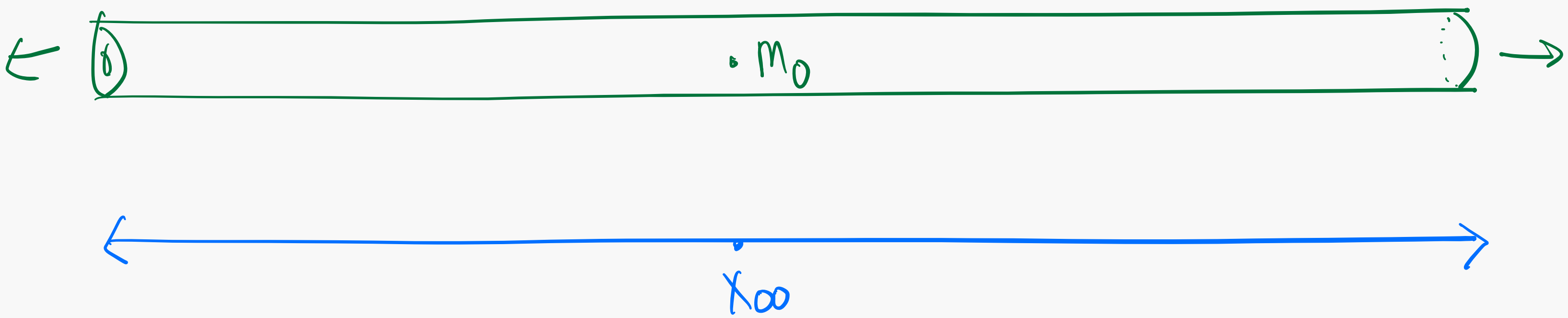
Suppose that there is no sequence $\{s_i\}_{i=1}^{\infty}$ so that $(M, \widehat{d}_{s_i,1}, m_0)$ has a three dimensional limit. Then for large s, the time-one slice $(M, \widehat{d}_{s,1}, m_0)$ of the rescaled Ricci flow is increasing Gromov-Hausdorff close to a pointed noncompact Alexandrov space (X_∞, x_∞) that is one dimensional or two dimensional.

The possibilities are that

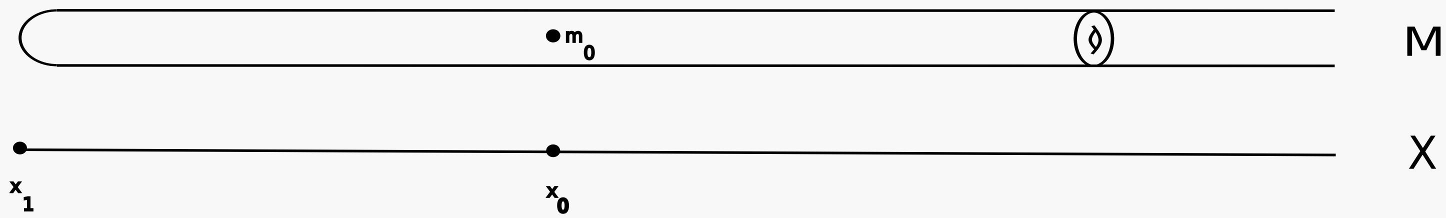
- X_∞ is a line.
- X_∞ is a ray.
- X_∞ is an Alexandrov surface.

Ricci flow II

From the theory of bounded curvature collapse (Cheeger-Fukaya-Gromov):

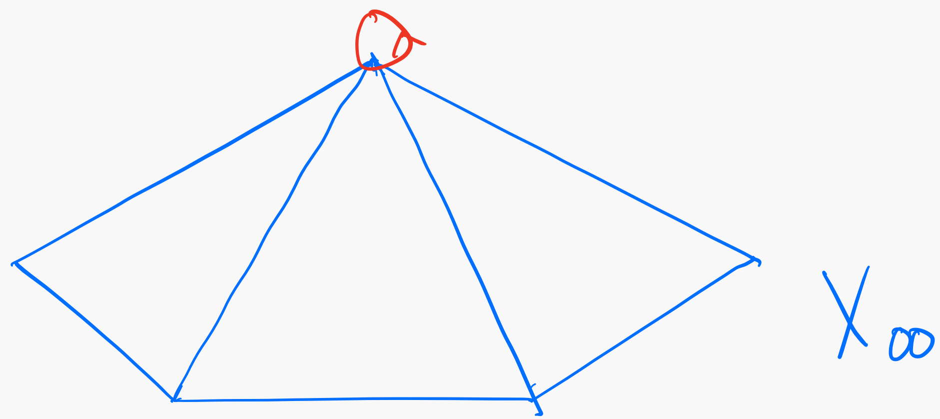


If X_∞ is a line then a large region of $(M, \widehat{d}_{s,1})$ around m_0 is a fiber bundle over an interval, with T^2-fiber.



If X_∞ is a ray then depending on the distance from x_∞ to the tip of the ray, a large region of $(M, \widehat{d}_{s,1})$ around m_0 is a fiber bundle over an interval, with T^2-fiber, or is part of a solid torus that (singular) fibers over an interval.

Ricci flow II



If X_∞ is an Alexandrov surface then a large region of $(M, \widehat{d}_{s,1})$ around m_0 is the total space of a Seifert bundle.

In any case, there is a short loop γ through m_0 that does not contract to a point in the unit ball $B_{\widehat{d}_{s,1}}(m_0, 1)$.

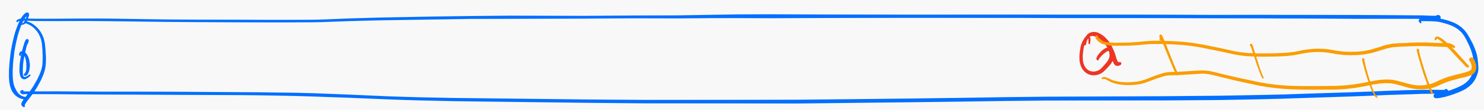
Ricci flow II

Because M is diffeomorphic to $\mathbb{R}^3$, there is some $\Delta \gg 0$ so that γ contracts to a point in the ball $B_{\widehat{d}_{s,1}}(m_0, \Delta)$ of radius Δ.

If we now start to increase s, distances decrease.

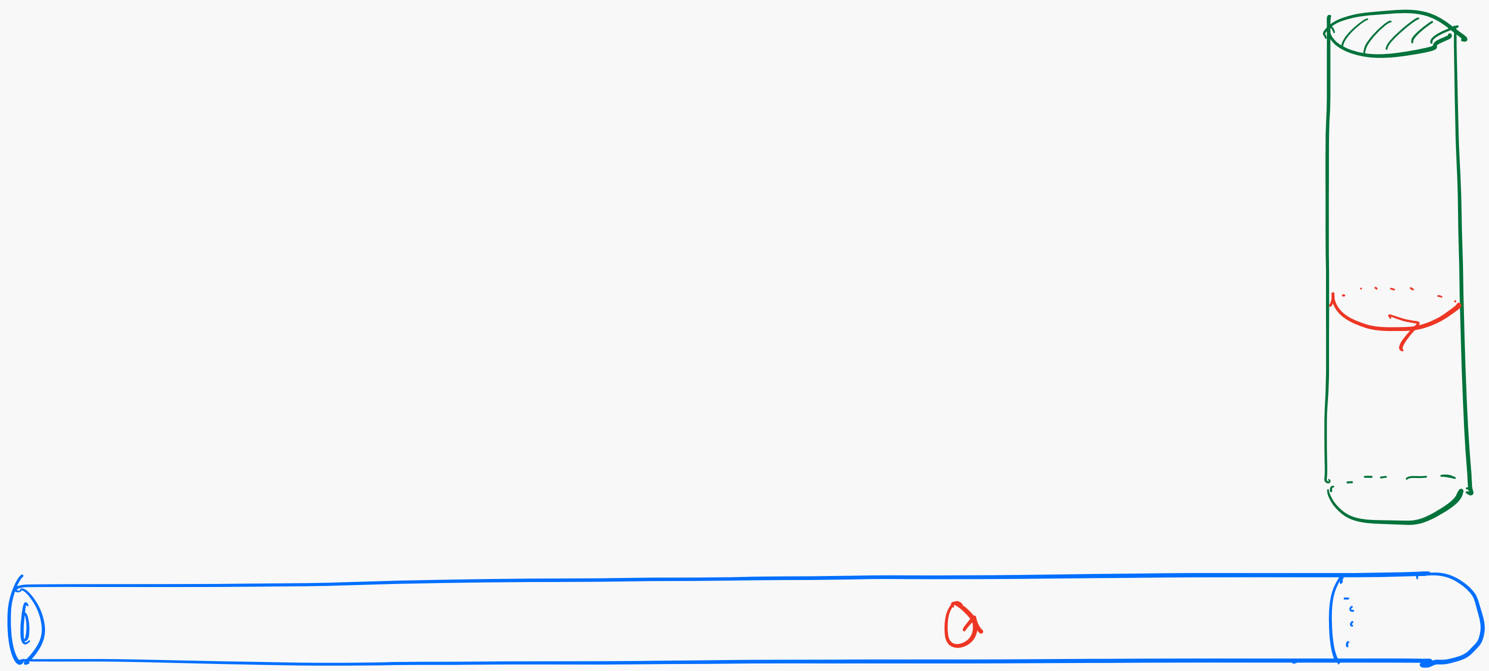
There will be some critical $\widehat{s}$ so that γ contracts in the unit ball $B_{\widehat{d}_{s,1}}(m_0, 1)$ if and only if $s > \widehat{s}$.

Ricci flow II



At the critical value $\widehat{s}$, the ball $B_{\widehat{d}_{\widehat{s},1}}(m_0, 1)$ must be Gromov-Hausdorff close to a ray whose tip has distance approximately one from x_∞. The loop γ through m_0 is even shorter than before.

Ricci flow II



Punchline: a very short loop through m_0 cannot contract in such a solid torus that is very close to an interval, under a uniform sectional curvature bound, unless it already contracts near m_0.

This is a contradiction.

Ricci flow II

Corollary: *Let (M, g) be a complete noncompact Riemannian 3-manifold having bounded curvature and positive c-pinched Ricci curvature.*

Then (M, g) has cubic volume growth, i.e.

$$\liminf_{r \to \infty} r^{-3} \operatorname{vol}(B(m_0, r)) > 0.$$

This follows from the fact that the blowdown limit is three dimensional, along with the distance distortion estimates.