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# A spinorial quasilocal mass (EP)

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**AFFILIATIONS**

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**ABSTRACT**

We define a quasilocal energy of a compact manifold-with-boundary, relative to a background manifold. The construction uses spinors on one manifold and the pullback of dual spinors from the other manifold. We prove positivity results for the quasilocal energy, in both the Riemannian and Lorentzian settings.

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## I. INTRODUCTION

The mass of an asymptotically flat initial data set is a quantity computed from the asymptotic geometry. It has the physical interpretation of measuring the gravitational energy within the space. The positive mass theorem says that under an appropriate positivity assumption on the curvature, the mass is positive.[1,2]

It is an old problem to localize the measurement of gravitational energy to a compact region, i.e. to give a good notion of a quasilocal mass. Some survey articles are Refs. 3 and 4. In most of this paper we will refer more precisely to a quasilocal energy rather than a quasilocal mass.

In view of Witten's proof of the positive energy theorem for spin manifolds,[2] an attractive approach to define a quasilocal energy is to use spinors. This idea has been considered by various people, notably Dougan–Mason[5] and Zhang.[6,7] There are two basic issues with such an approach. First, the quasilocal energy is always defined relative to some background space. For example, in the positive energy theorem, the background space is Euclidean space. It is not clear how to incorporate the background space into the definition of the quasilocal energy. The second issue is what boundary conditions to impose on the spinor fields. Witten's proof used spinor fields that asymptotically approach a constant spinor with respect to the model flat space. It is not evident what the analog should be for a compact manifold-with-boundary.

We show that both of these issues can be handled by the technique of pulling back (dual) spinors from the background space. This technique has some history in the mathematical literature and has been applied to prove sharp comparison results about scalar curvature by Llarull,[8,9] Goette-Semmelmann,[10] the author,[11] and Wang–Xie–Yu.[12] One advantage of the spinorial approach to the quasilocal energy is that the nonnegativity of the energy, under some curvature conditions, falls out immediately.

In Sec. II we define a quasilocal energy in the Riemannian setting. Given compact connected $n$-dimensional Riemannian manifolds-with-boundary $(N, \partial N)$ and $(M, \partial M)$, suppose that $f : (N, \partial N) \to (M, \partial M)$ is a spin map that is an isometry on $\partial N$. We think of $M$ as the background space and $N$ as the manifold whose quasilocal energy we want to define, relative to $M$. There is a corresponding Clifford module $E$ on $N$. For simplicity, suppose that $n$ is even dimensional and that $N$ and $M$ are spin, in which case $E$ is the twisted spinor bundle $S_N \otimes f^* S_M^*$. Let $D^N$ be the corresponding Dirac-type operator on $C^\infty(N; E)$.

On $\partial N$, we have the identifications

$$E\big|_{\partial N} \cong \left(S_N \otimes (\partial f)^* S_M^*\right)\big|_{\partial N} \cong \left(S_N \otimes S_N^*\right)\big|_{\partial N} \cong \Lambda^* T^* N\big|_{\partial N} \cong \Lambda^* T^* \partial N \oplus \Lambda^* T^* \partial N, \qquad (1.1)$$

where the last isomorphism is a separation into tangential forms and forms with a normal component. Let $\pi_+ : \Lambda^* T^* N\big|_{\partial N} \to \Lambda^* T^* \partial N$ be projection onto the tangential component. There is a natural "Dirichlet" boundary condition for sections $\psi$ of $E$ given by $\pi_+\left(\psi\big|_{\partial N}\right) = 0$; let $\mathcal{D}$ denote the ensuing self-adjoint operator. For simplicity, in this introduction we mostly consider the case when $\operatorname{Ker}(\mathcal{D}) = 0$.

To set up a boundary value problem, there is a canonical choice of section of $\Lambda^* T^* \partial N$, namely the constant function 1. Let $e_n$ be the inward-pointing unit normal on $\partial N$.

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*Definition 1. If $C$ is a nonempty union of connected components of $\partial N$, let $\psi \in C^\infty(N; E)$ be the (unique) solution to $D^N \psi = 0$ with $\pi_+ \left( \psi|_{\partial N} \right) = 1_C$, the characteristic function of $C$. The quasilocal energy associated to the boundary subset $C$ is*

$$\mathcal{E}_C = -\int_{\partial N} \langle \psi, \nabla^N_{e_n} \psi \rangle \, \mathrm{dvol}_{\partial N}. \tag{1.2}$$

Here are some basic properties of the quasilocal energy. Let $R_N$ and $R_M$ denote the scalar curvatures of $N$ and $M$, respectively. Define $|\Lambda^2 df|$, the distortion of $f$ on two-forms, to be the pointwise norm $\sup_{v \wedge w \neq 0} \frac{|df(v) \wedge df(w)|}{|v \wedge w|}$.

We recall that the boundary map $\partial f$ is an isometry on $\partial N$.

**Theorem 1.**

1. If $f$ is an isometric diffeomorphism then $\mathcal{E}_{\partial N} = 0$.
2. If $M$ has nonnegative curvature operator and $R_N \geq |\Lambda^2 df|(f^* R_M)$ then $\mathcal{E}_C \geq 0$.

Part 1 of Theorem 1 is consistent with the interpretation of $\mathcal{E}$ as being a relative energy between $N$ and $M$. Part 2 of Theorem 1 is an immediate consequence of a Bochner-type formula. In particular, if $M$ is flat and $R_N \geq 0$ then $\mathcal{E}_C \geq 0$.

Although (1.2) is an integral over the boundary of $N$, the solution $\psi$ to the boundary value problem depends *a priori* on the interior geometry of $N$ and the map $f$. When $M$ is a domain in $\mathbb{R}^n$, the quasilocal energy only depends on $f$ through its boundary restriction $\partial f$.

We give a formula for $\mathcal{E}_C$ indicating that in the weak field limit, i.e. when $N$ is a perturbation of $M$, the quasilocal energy is approximately equal to the Brown–York energy.[13] We expect, but do not prove, that if one takes an appropriate exhaustion of an asymptotically flat manifold by compact domains then their quasilocal energies will approach a normalization constant times the ADM mass. We compute $\mathcal{E}_{\partial N}$ when $N$ is conformally related to $M$ and find that it is exactly the Brown–York energy. We find a similar statement when $N$ is rotationally symmetric.

In Sec. III we consider a Lorentzian quasilocal energy. That is, we have Lorentzian $(n + 1)$-dimensional manifolds $\overline{N}$ and $\overline{M}$, along with compact connected spatial hypersurfaces-with-boundary $N \subset \overline{N}$ and $M \subset \overline{M}$. Again, we use a comparison map $f : (N, \partial N) \to (M, \partial M)$ that is an isometry on $\partial N$ and look at the Dirac-type operator $D^N$ acting on sections of the twisted spinor bundle $S_N \otimes f^* S_M^*$. There is a new feature that, in general, $D^N$ need not be formally self-adjoint on the interior of $N$. We first look at the case when $M$ is a totally geodesic hypersurface in $\overline{M}$. Then there is no problem with self-adjointness and results from Sec. II extend. The curvature condition in part 2 of Theorem 1 gets replaced by $2\left( T_{00} - \sqrt{-\sum_{\alpha=1}^n T_{0\alpha} T^{0\alpha}} \right) \geq |\Lambda^2 df|(f^* R_M)$, where $T_{AB} = R^{\overline{N}}_{AB} - \frac{1}{2} R_{\overline{N}} g^{\overline{N}}_{AB}$ is the Einstein tensor of $\overline{N}$.

We next look at the case when $\overline{M}$ is the flat Minkowski space $\mathbb{R}^{n,1}$, but the hypersurface $M$ need not be totally geodesic. It turns out that $D^N$ is formally self-adjoint on the interior of $N$ if one uses an appropriate weighted $L^2$-space. Regarding boundary conditions, there are two natural choices. The first one does not give a self-adjoint boundary value problem but nevertheless one can use it to define a quasilocal energy $\mathcal{E}_C$. The second one does give a self-adjoint boundary value problem and gives rise to a quasilocal energy $\mathcal{E}^{sa}_C$. They have the following properties.

**Theorem 2.**

1. If $f$ is an isometric diffeomorphism then $\mathcal{E}_{\partial N} = 0$.
2. If $\overline{N}$ satisfies the dominant energy condition $T_{00} \geq \sqrt{-\sum_{\alpha=1}^n T_{0\alpha} T^{0\alpha}}$ then $\mathcal{E}_C \geq 0$ and $\mathcal{E}^{sa}_C \geq 0$.

Comparing $\mathcal{E}_C$ and $\mathcal{E}^{sa}_C$, the first one $\mathcal{E}_C$ has the advantage, from part 1 of Theorem 2, of vanishing when $C = \partial N$ and $f$ is an isometric diffeomorphism. On the other hand, $\mathcal{E}^{sa}_C$ has the advantage of coming from a self-adjoint boundary value problem.

Finally, we extend the results to the case when $\overline{M}$ is a more general Lorentzian manifold than Minkowski space. It turns out that it is enough for $\overline{M}$ to be a product spacetime $\mathbb{R} \times X$ for some Riemannian manifold $X$. This can be compared with Chen *et al.*'s construction of a quasilocal energy when the background space is in a static spacetime.[14]

Regarding earlier work about a spinorial approach to quasilocal energy, Dougan and Mason used the complex structure on an oriented surface to make an interesting choice of boundary condition.[5] Their approach is clearly restricted to $n = 3$. Zhang used the pure Dirac operator to define a quasilocal energy when the background space $M$ is a domain in $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$.[6] His boundary condition came from using constant spinors on $\mathbb{R}^{3,1}$; compare with Proposition 12 of the present paper. He imposed geometric restrictions to ensure that the Dirac operator is invertible; compare with Proposition 11 of the present paper. In Ref. 7 he also allowed $M$ to be a domain in a hyperbolic submanifold of $\mathbb{R}^{3,1}$.

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## II. RIEMANNIAN CASE

This section is about the quasilocal energy in the Riemannian case. We begin with even dimensional spaces. Section II A has background information. Section II B has the definition of the quasilocal energy. In Sec. II C we prove its basic properties. Section II D computes the quasilocal energy for conformally related manifolds. Section II E specializes to when the background space $M$ is a domain in $\mathbb{R}^n$ and also treats rotationally symmetric manifolds $N$. Finally, Sec. II F covers the odd dimensional case.

## A. Background

Let $R$ denote scalar curvature and let $H$ denote mean curvature. With our convention, $\partial B^n$ has $H = n - 1$.

Let $N$ and $M$ be compact connected $n$-dimensional Riemannian manifolds with nonempty boundary. Let $f : (N, \partial N) \to (M, \partial M)$ be a smooth spin map, i.e. $TN \oplus f^* TM$ admits a spin structure. Equivalently, $f^* w_2(M) = w_2(N)$. Let $\partial f : \partial N \to \partial M$ denote the restriction to the boundary. We assume that for each component $Z$ of $\partial N$, the restriction $\partial f|_Z$ is an isometric diffeomorphism from $Z$ to $f(Z)$.

For simplicity, we assume that $N$ and $M$ are spin and that $\partial f$ is a spin diffeomorphism; the general case is similar. We assume first that $n$ is even. Then a spinor representation $S^{\pm}$ of $\mathrm{Spin}(n)$ has complex dimension $2^{\frac{n}{2}-1}$, while the Clifford algebra has a faithful representation space $S = S^+ \oplus S^-$ of complex dimension $2^{\frac{n}{2}}$. We let $S^*$ denote the complex vector space of complex-linear functionals on $S$, i.e. no complex conjugation involved. Let $S_N$ denote the spinor bundle on $N$, and similarly for $S_M$. Put $E = S_N \otimes f^* S_M^*$, a Clifford module on $N$. (This Clifford module exists in the general case.) We take the inner product $\langle \cdot, \cdot \rangle$ on $E$ to be $\mathbb{C}$-linear in the second slot and $\mathbb{C}$-antilinear in the first slot. As $\mathrm{End}(S_N) \cong \Lambda^*(T^*N)$, we can identify $E|_{\partial N}$ with $\Lambda^*(T^*N)\big|_{\partial N}$ (We require this property in the general case, which amounts to a choice of spin structure on $E|_{\partial N}$.).

Let $\{e_\alpha\}_{\alpha=1}^n$ be a local oriented orthonormal framing on $N$, with dual basis $\{\tau^\alpha\}_{\alpha=1}^n$. Let $\omega^\alpha{}_{\beta\gamma}$ be the connection one-forms with respect to $\{e_\alpha\}_{\alpha=1}^n$. Let $\{\widehat{e}_{\widehat{\alpha}}\}_{\widehat{\alpha}=1}^n$ of $M$ be a local oriented orthonormal framing of $M$. Let $\widehat{\omega}^{\widehat{\alpha}}{}_{\widehat{\beta}\gamma}$ be the pullbacks under $f$ of the connection one-forms with respect to $\{\widehat{e}_{\widehat{\alpha}}\}_{\widehat{\alpha}=1}^n$.

To make things more symmetric, it will be convenient to do local calculations on $S_N \otimes f^* S_M$ rather than $S_N \otimes f^* S_M^*$. In the Riemannian setting, $S_M^*$ is unitarily equivalent to $S_M$ and so we don't lose anything this way. Let $\{\gamma^\alpha\}_{\alpha=1}^n$ be Clifford multiplication by $\{e_\alpha\}_{\alpha=1}^n$, satisfying $\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\delta^{\alpha\beta}$. Let $\{\widehat{\gamma}^{\widehat{\alpha}}\}_{\widehat{\alpha}=1}^n$ be Clifford multiplication by $\{\widehat{e}_{\widehat{\alpha}}\}_{\widehat{\alpha}=1}^n$. They both have an odd grading, in the sense that in calculations we take $\gamma^\alpha$ to anticommute with $\widehat{\gamma}^{\widehat{\alpha}}$.

The covariant derivative on $E$ has the local form

$$\nabla^N_\sigma = e_\sigma + \frac{1}{8}\omega_{\alpha\beta\sigma}[\gamma^\alpha, \gamma^\beta] + \frac{1}{8}\widehat{\omega}_{\widehat{\alpha}\widehat{\beta}\sigma}[\widehat{\gamma}^{\widehat{\alpha}}, \widehat{\gamma}^{\widehat{\beta}}]. \tag{2.1}$$

The Dirac operator on $C^\infty(N; E)$ is $D^N = -\sqrt{-1}\sum_{\sigma=1}^n \gamma^\sigma \nabla^N_\sigma$.

We will take the orthonormal frame $\{e_\alpha\}$ at a point in $\partial N$ so that $e_n$ is the inward-pointing unit normal vector there. Let $\mathrm{dvol}_N$ denote the Riemannian density on $N$, and similarly for $\mathrm{dvol}_{\partial N}$. Given $\psi_1, \psi_2 \in C^\infty(N; E)$, we have

$$\int_N \langle D^N \psi_1, \psi_2 \rangle \mathrm{dvol}_N - \int_N \langle \psi_1, D^N \psi_2 \rangle \mathrm{dvol}_N = -\sqrt{-1}\int_{\partial N} \langle \psi_1, \gamma^n \psi_2 \rangle \mathrm{dvol}_{\partial N}. \tag{2.2}$$

The Lichnerowicz formula implies

$$(D^N)^2 = (\nabla^N)^* \nabla^N + \frac{R_N}{4} - \frac{1}{4}[\gamma^\sigma, \gamma^\tau]\left(\frac{1}{8}\widehat{R}_{\widehat{\alpha}\widehat{\beta}\sigma\tau}[\widehat{\gamma}^{\widehat{\alpha}}, \widehat{\gamma}^{\widehat{\beta}}]\right). \tag{2.3}$$

We now extend some computations in Ref. 15, Proof of Lemma 4.1. Suppose that $D^N \psi = 0$. In what follows, summations over Greek letters will go from 1 to $n$ and summations over Latin letters will go from 1 to $n - 1$. Equation (2.3) implies that

$$0 = \int_N |\nabla^N \psi|^2 \, \mathrm{dvol}_N + \int_{\partial N} \langle \psi, \nabla^N_{e_n}\psi \rangle \mathrm{dvol}_{\partial N} + \frac{1}{4}\int_N R_N |\psi|^2 \, \mathrm{dvol}_N - \tag{2.4}$$

$$\frac{1}{32}\int_N \widehat{R}_{\widehat{\alpha}\widehat{\beta}\sigma\tau}\langle \psi, [\gamma^\sigma, \gamma^\tau][\widehat{\gamma}^{\widehat{\alpha}}, \widehat{\gamma}^{\widehat{\beta}}]\psi \rangle.$$

Now $D^N \psi = 0$ implies that on $\partial N$, we have

$$\nabla^N_{e_n}\psi = -\gamma^n \sum_{i=1}^{n-1} \gamma^i \nabla^N_i \psi \tag{2.5}$$

$$= -\gamma^n \sum_{i=1}^{n-1} \gamma^i \left(\nabla^{\partial N}_i \psi + \frac{1}{2}\omega_{nji}\gamma^n \gamma^j \psi + \frac{1}{2}\widehat{\omega}_{\widehat{n}\widehat{j}i}\widehat{\gamma}^{\widehat{n}}\widehat{\gamma}^{\widehat{j}}\psi\right)$$

$$= D^{\partial N}\psi + \frac{H_{\partial N}}{2}\psi - \frac{1}{2}\gamma^n \gamma^i \widehat{\gamma}^{\widehat{n}}\widehat{\gamma}^{\widehat{j}}\widehat{A}_{\widehat{j}i}\psi,$$

where

$$D^{\partial N} = -\gamma^n \sum_{i=1}^{n-1} \gamma^i \nabla^{\partial N}_i \tag{2.6}$$

is the Dirac operator on $\partial N$ coupled to $(\partial f)^* S_M^*$, $\widehat{A}$ is the second fundamental form of $M$ and $\widehat{A}_{\widehat{j}iD} = \widehat{A}(\widehat{e}_{\widehat{j}}, (\partial f)_*(e_i))$.

*Remark 1.* The factor of $\frac{1}{2}$ in (2.5) corrects the $\frac{1}{4}$ that appears in Refs. 11 and 15.

From Ref. 10, Sec. 1.1, if $M$ has nonnegative curvature operator then

$$\frac{1}{32}\widehat{R}_{\widehat{\alpha}\widehat{\beta}\sigma\tau}[\gamma^\sigma, \gamma^\tau][\widehat{\gamma}^{\widehat{\alpha}}, \widehat{\gamma}^{\widehat{\beta}}] \le \frac{1}{4}|\Lambda^2 df|(f^* R_M)\mathrm{Id}_E. \tag{2.7}$$

[The paper[10] assumes that $|\Lambda^2 df| \le 1$ but their argument shows the result stated in (2.7); cf. Ref. 16].

*Proposition 1.* If $M$ has nonnegative curvature operator and $R_N \ge |\Lambda^2(df)|(f^* R_M)$ then for any $\psi \in C^\infty(N; E)$, we have $-\int_{\partial N} \langle \psi, \nabla^N_{e_n}\psi\rangle dvol_{\partial N} \ge 0$. If $M$ has nonnegative curvature operator, $R_N \ge |\Lambda^2(df)|(f^* R_M)$, $\psi$ is nonzero and $-\int_{\partial N} \langle \psi, \nabla^N_{e_n}\psi\rangle dvol_{\partial N} = 0$ then $R_N = |\Lambda^2(df)|(f^* R_M)$.

*Proof.* This follows from (2.4) and (2.7). □

## B. Definition of the quasilocal energy

Let $\epsilon$ be the $\mathbb{Z}_2$-grading operator on $S_N$. Put $\gamma^0 = i\epsilon$. Let $T$ be the involution on $S_N|_{\partial N}$ given by

$$T\eta = \gamma^0 \gamma^n \eta. \tag{2.8}$$

Identifying $f^* S_M^*|_{\partial N}$ with $S_N^*|_{\partial N}$, there is an induced involution $T$ on

$$\Lambda^* T^* N|_{\partial N} \cong \mathrm{End}(S_N)|_{\partial N} \cong (S_N \otimes S_N^*)|_{\partial N} \cong S_N|_{\partial N} \otimes f^* S_M^*|_{\partial M}. \tag{2.9}$$

We recall that an element of $\Lambda^* T^* N$ acts on $S_N$ by Clifford multiplication, realizing the isomorphism $\Lambda^* T^* N \cong \mathrm{End}(S_N)$. Then for $\omega \in \Lambda^* T^* N|_{\partial N}$ and $\eta \in S_N|_{\partial N}$, we have $T(\omega \cdot \eta) = T(\omega) \cdot T(\eta)$. In terms of the orthonormal frame, $T$ acts on $T^* N|_{\partial N}$ as $\gamma^0 \gamma^n \widehat{\gamma}^{\widehat{0}} \widehat{\gamma}^{\widehat{n}}$.

Let $\tau^n$ be the dual covector to $e_n$.

*Lemma 1.* Given $\omega \in \Lambda^* T^* N|_{\partial N}$, write $\omega = \omega_+ + \tau^n \wedge \omega_-$ for $\omega_\pm \in \Lambda^* T^* \partial N$. Then $T\omega = \omega_+ - \tau^n \wedge \omega_-$.

*Proof.* Given an increasing multi-index $I$ with entries between 1 and $n - 1$, and $\psi \in S_N|_{\partial N}$, we have

$$T(\gamma^I \psi) = (\gamma^0 \gamma^n)\gamma^I \psi = \gamma^I(\gamma^0 \gamma^n)\psi = \gamma^I T(\psi) \tag{2.10}$$

and

$$T(\gamma^n \gamma^I \psi) = (\gamma^0 \gamma^n)\gamma^n \gamma^I \psi = -\gamma^n \gamma^I (\gamma^0 \gamma^n)\psi = -\gamma^n \gamma^I T(\psi) \tag{2.11}$$

Using the isomorphism $\Lambda^* T^* N|_{\partial N} \cong \mathrm{End}(S_N)|_{\partial N}$, the lemma follows. □

Let $\pi_\pm$ denote orthogonal projection onto the $\pm 1$-eigenspace of $T$, acting on $\Lambda^* T^* N|_{\partial N}$, so $\mathrm{Im}(\pi_+)$ is isomorphic to $\Lambda^* T^* \partial N$.

Consider the operator $D^N$ acting on elements $\psi \in C^\infty(N; E)$. The boundary condition $\pi_+(\psi|_{\partial N}) = 0$, which is an analog of Dirichlet boundary conditions, defines an elliptic boundary condition for $D^N$ (Ref. 17, Sec. 7.5). One can check that with the given boundary condition, $D^N$ is formally self-adjoint. Let $\mathcal{D}$ denote the ensuing self-adjoint operator, densely defined on $L^2(N; E)$ (Ref. 17, Chap. 7). It has compact resolvent. The next proposition gives a sufficient condition for $\mathcal{D}$ to be invertible.

*Proposition 2.* Suppose that

- $M$ has nonnegative curvature operator,
- $\partial M$ has nonnegative second fundamental form,
- $R_N \ge |\Lambda^2 df|(f^* R_M)$,
- $H_{\partial N} \ge (\partial f) * H_{\partial M}$, and
- $R_N > |\Lambda^2 df|(f^* R_M)$ somewhere or $H_{\partial N} > (\partial f) * H_{\partial M}$ somewhere.
  Then $\mathrm{Ker}(\mathcal{D}) = 0$.

*Proof.* Suppose that $\psi \in \mathrm{Ker}(\mathcal{D})$. From (2.4) and (2.5),

$$0 = \int_N |\nabla^N \psi|^2 \, dvol_N + \frac{1}{4}\int_N R_N|\psi|^2 \, dvol_N - \frac{1}{32}\int_N \widehat{R}_{\widehat{\alpha}\widehat{\beta}\sigma\tau}\langle\psi, [\gamma^\sigma, \gamma^\tau][\widehat{\gamma}^{\widehat{\alpha}}, \widehat{\gamma}^{\widehat{\beta}}]\psi\rangle \tag{2.12}$$
$$+ \int_{\partial N} \langle\psi, D^{\partial N}\psi\rangle \, dvol_{\partial N} + \frac{1}{2}\int_{\partial N} H_{\partial N}|\psi|^2 \, dvol_{\partial N} - \frac{1}{2}\int_{\partial N} \langle\psi, \gamma^n\gamma^i\widehat{\gamma}^{\widehat{n}}\widehat{\gamma}^{\widehat{j}}\widehat{A}_{\widehat{j}i}\psi\rangle \, dvol_{\partial N}.$$

As $T$ anticommutes with $D^{\partial N}$, we have

$$\langle \psi, D^{\partial N}\psi \rangle = -\langle \psi, D^{\partial N}T\psi \rangle = \langle \psi, TD^{\partial N}\psi \rangle = \langle T\psi, D^{\partial N}\psi \rangle = -\langle \psi, D^{\partial N}\psi \rangle, \tag{2.13}$$

so $\langle \psi, D^{\partial N}\psi \rangle = 0$. From Ref. 11, Lemma 2.1, $\langle \psi, \gamma^n \gamma^i \widehat{\gamma}^n \widehat{\gamma}^j \widehat{A_{ji}} \psi \rangle \leq (\partial f)^* H_{\partial M} |\psi|^2$. Then (2.7) and (2.12) imply that $\nabla^N \psi = 0$. Since $\nabla^N$ is unitary, it follows that $|\psi|^2$ is constant on the connected manifold $N$. Then (2.12) implies that $|\psi|^2 = 0$. □

*Remark 2.* If the Euler characteristic of $M$ is nonzero and the degree of $f$ is nonzero then $\mathrm{Ker}(\mathcal{D}) \neq 0$ (Ref. 11, Sec. 2.2).

We now discuss the boundary value problem. Given $\sigma \in \Omega^*(\partial N)$ with $\pi_+(\sigma) = \sigma$, we wish to find some $\psi \in C^\infty(N; E)$ so that $D^N\psi = 0$ and $\pi_+(\psi|_{\partial N}) = \sigma$. Suppose that we can do this. If $\eta \in \mathrm{Ker}(\mathcal{D})$ then from (2.2), we see that $\int_{\partial N} \langle \sigma, \gamma^n \eta \rangle \mathrm{dvol}_{\partial N} = 0$ [Note that $\eta|_{\partial N} \in \mathrm{Im}(\pi_-)$, so $\gamma^n \eta|_{\partial N} \in \mathrm{Im}(\pi_+)$.].

Conversely, given $\sigma \in \Omega^*(\partial N)$ with $\pi_+(\sigma) = \sigma$, if $\int_{\partial N} \langle \sigma, \gamma^n \eta \rangle \mathrm{dvol}_{\partial N} = 0$ for all $\eta \in \mathrm{Ker}(\mathcal{D})$ then there is some $\psi \in C^\infty(N; E)$ so that $D^N\psi = 0$ and $\pi_+(\psi|_{\partial N}) = \sigma$ (Ref. 18, Lemma 14). The section $\psi$ is unique up to addition by elements of $\mathrm{Ker}(\mathcal{D})$.

Let $C$ be a nonempty union of connected components of $\partial N$. To define the quasilocal energy, we would like to take $\sigma$ to be the characteristic function $1_C$. To be sure that $\psi$ exists, we may have to slightly modify this choice. Let $P_{\gamma^n \mathrm{Ker}(\mathcal{D})|_{\partial N}}$ be orthogonal projection from $\Omega^*(\partial N)$ to $\gamma^n \mathrm{Ker}(\mathcal{D})|_{\partial N}$. We put $\overline{1}_C = 1_C - P_{\gamma^n \mathrm{Ker}(\mathcal{D})|_{\partial N}} 1_C$. We then take $\sigma = \overline{1}_C$ and construct $\psi$ accordingly.

*Definition 2.* Put

$$\mathcal{C}_C = \left\{ \psi \in C^\infty(N;E) \ : \ D^N\psi = 0, \pi_+(\psi|_{\partial N}) = \overline{1}_C \right\}. \tag{2.14}$$

*The quasilocal energy associated to the boundary subset $C$ is*

$$\mathcal{E}_C = \inf_{\psi \in \mathcal{C}_C} \left( -\int_{\partial N} \langle \psi, \nabla^N_{e_n} \psi \rangle \, \mathrm{dvol}_{\partial N} \right). \tag{2.15}$$

Note that $\mathcal{E}_C$ could be $-\infty$.

*Lemma 2.* If $\mathrm{Ker}(\mathcal{D}) = 0$ then $\mathcal{E}_C > -\infty$. In general, if $\mathcal{E}_C > -\infty$ then it is realized by some $\psi_{min} \in \mathcal{C}_C$.

*Proof.* We know that $\mathcal{C}_C$ is a finite dimensional affine space modelled on $\mathrm{Ker}(\mathcal{D})$. If $\mathrm{Ker}(\mathcal{D}) = 0$ then $\mathcal{C}_C$ has a single element and $\mathcal{E}_C > -\infty$. If $\mathrm{Ker}(\mathcal{D}) \neq 0$ then as $-\int_{\partial N} \langle \psi, \nabla^N_{e_n} \psi \rangle \, \mathrm{dvol}_{\partial N}$ is a quadratic function on $\mathcal{C}_C$, the lemma follows. □

In some statements that follow we may implicitly assume that $\mathcal{E}_C > -\infty$. We will see relevant examples where this is the case in Propositions 4, 5, Lemma 4, and Proposition 9.

## C. Properties of the quasilocal energy

*Proposition 3.* If $f : N \to M$ is an isometric diffeomorphism then $\mathcal{E}_{\partial N} = 0$.

*Proof.* If $f$ is an isometric diffeomorphism then we can take $E \cong \Lambda^* T^* N$ and identify $D^N$ with $d + d^*$. Letting $*$ denote the Hodge duality operator, if $\psi \in C^\infty(N; E)$ then

$$\begin{aligned}
\int_N \langle (D^N)^2 \psi, \psi \rangle \, \mathrm{dvol}_N &= \int_N \langle (dd^* + d^*d)\psi, \psi \rangle \, \mathrm{dvol}_N \\
&= \int_N \left( dd^*\psi \wedge *\psi + d^*d\psi \wedge *\psi \right) \\
&= \int_N \left( dd^*\psi \wedge *\psi + *d^*d\psi \wedge *(*\psi) \right) \\
&= \int_N \left( dd^*\psi \wedge *\psi + dd^* *\psi \wedge *(*\psi) \right) \\
&= \int_N \left( |d^*\psi|^2 + |d\psi|^2 \right) \mathrm{dvol}_N + \int_{\partial N} \left( d^*\psi \wedge *\psi + d^* *\psi \wedge *(*\psi) \right).
\end{aligned} \tag{2.16}$$

Comparing with Eq. (2.4) shows

$$-\int_{\partial N} \langle \psi, \nabla^N_{e_n} \psi \rangle \, \mathrm{dvol}_{\partial N} = -\int_{\partial N} \left( d^*\psi \wedge *\psi + d^* *\psi \wedge *(*\psi) \right), \tag{2.17}$$

There is a solution $\psi$ of $D^N\psi = 0$ given by $\psi = 1$, with $\pi_+(1) = 1$. It follows that 1 is orthogonal to $\gamma^n \mathrm{Ker}(\mathcal{D})|_{\partial N}$, so $\overline{1}_{\partial N} = 1$.

If $i : \partial N \to N$ is the boundary inclusion then an element of $\mathrm{Ker}(\mathcal{D})$ is a differential form $\rho \in \Omega^*(N)$ satisfying $(d + d^*)\rho = 0$ and the relative (Dirichlet) boundary condition $i^*\rho = 0$. Applying $i^*$, it follows that $i^*(d^*\rho) = 0$. Squaring $d + d^*$, it follows that $(dd^* + d^*d)\rho = 0$. Hence $\rho$ is a harmonic form $\alpha$ on $N$ satisfying relative (Dirichlet) boundary conditions. In particular, $d\rho = d^*\rho = 0$.

Hence elements $\psi$ of $\mathcal{C}_{\partial N}$ are of the form $1 + \rho$ for such $\rho$. As $d\psi = d^*\psi = 0$, Eq. (2.17) gives $-\int_{\partial N} \langle \psi, \nabla^N_{e_n} \psi \rangle \, \mathrm{dvol}_{\partial N} = 0$, which proves the lemma. $\qquad \square$

We now give the basic positivity property of the quasilocal energy in the physically relevant case.

**Proposition 4.** *Suppose that $M$ has nonnegative curvature operator. If $R_N \geq |\Lambda^2 df|(f^* R_M)$ then $\mathcal{E}_C \geq 0$. If $R_N \geq |\Lambda^2 df|(f^* R_M)$ and $\mathcal{E}_C = 0$ then $R_N = |\Lambda^2 df|(f^* R_M)$.*

*Proof.* This follows from (2.4) and (2.7). $\qquad \square$

**Proposition 5.** *If $M$ is flat and $R_N \geq 0$ then $\mathcal{E}_C \geq 0$. If $M$ is flat, $R_N \geq 0$ and $\mathcal{E}_C = 0$ then $N$ is Ricci flat.*

*Proof.* The first statement follows from Proposition 4. The second statement follows as in Ref. 11, Proof of Proposition 2.3. $\qquad \square$

We now write $\mathcal{E}_C$ more explicitly as a boundary integral. From (2.5), we have

$$\mathcal{E}_C = -\int_{\partial N} \langle \psi_{min}, D^{\partial N} \psi_{min} \rangle \, \mathrm{dvol}_{\partial N} - \frac{1}{2} \int_{\partial N} H_{\partial N} |\psi_{min}|^2 \, \mathrm{dvol}_{\partial N} \tag{2.18}$$
$$+ \frac{1}{2} \int_{\partial N} \langle \psi_{min}, \gamma^n \gamma^i \widehat{\gamma}^n \widehat{\gamma}^j \widehat{A}_{ji} \psi_{min} \rangle \, \mathrm{dvol}_{\partial N}.$$

We write $\psi_{min}|_{\partial N} = \overline{1}_C + \tau^n \wedge \phi$ for some $\phi \in \Omega^*(\partial N)$. Let $E^i$ denote exterior multiplication by $\tau^i$ and $I^i$ denote interior multiplication by $e_i$, when acting on $\Lambda^* T^* \partial N + \tau^n \wedge \Lambda^* T^* \partial N$.

**Proposition 6.** *If $\overline{1}_C = 1_C$ then*

$$\mathcal{E}_C = -\frac{1}{2} \int_C H_{\partial N} \, \mathrm{dvol}_{\partial N} + \frac{1}{2} \int_C (\partial f)^* H_{\partial M} \, \mathrm{dvol}_{\partial N} \tag{2.19}$$
$$- \frac{1}{2} \int_{\partial N} H_{\partial N} |\phi|^2 \, \mathrm{dvol}_{\partial N} + \int_{\partial N} \frac{1}{2} \widehat{A}_{ji} \langle \phi, (E^i - I^i)(E^j + I^j)\phi \rangle \, \mathrm{dvol}_{\partial N}.$$

*Proof.* On $\partial N$, we can represent $\gamma^i$ by $\sqrt{-1}(E^i - I^i)$, $\gamma^n$ by $\sqrt{-1}(E^n - I^n)$, $\widehat{\gamma}^i$ by $E^i + I^i$ and $\widehat{\gamma}^n$ by $E^n + I^n$.

**Lemma 3.** *We have*

$$\int_{\partial N} \langle \psi_{min}, D^{\partial N} \psi_{min} \rangle \, \mathrm{dvol}_{\partial N} = 0. \tag{2.20}$$

*Proof.* As $D^{\partial N}$ anticommutes with $T$, it follows that

$$\int_{\partial N} \langle 1_C, D^{\partial N} 1_C \rangle \, \mathrm{dvol}_{\partial N} = \int_{\partial N} \langle \tau^n \wedge \phi, D^{\partial N}(\tau^n \wedge \phi) \rangle \, \mathrm{dvol}_{\partial N} = 0. \tag{2.21}$$

Next, since $\nabla^{\partial N}$ restricts to the Riemannian connection on $\Lambda^* T^* \partial N$, we know that $\nabla^{\partial N} 1_C = 0$, so

$$\int_{\partial N} \langle \tau^n \wedge \phi, D^{\partial N} 1_C \rangle \, \mathrm{dvol}_{\partial N} = 0 \tag{2.22}$$

and

$$\int_{\partial N} \langle 1_C, D^{\partial N} \tau^n \wedge \phi \rangle \, \mathrm{dvol}_{\partial N} = \int_{\partial N} \langle D^{\partial N} 1_C, \tau^n \wedge \phi \rangle \, \mathrm{dvol}_{\partial N} = 0 \tag{2.23}$$

The lemma follows. $\qquad \square$

Also

$$\int_{\partial N} H_{\partial N} |\psi_{min}|^2 \, \mathrm{dvol}_{\partial N} = \int_C H_{\partial N} \, \mathrm{dvol}_{\partial N} + \int_{\partial N} H_{\partial N} |\phi|^2 \, \mathrm{dvol}_{\partial N}. \tag{2.24}$$

Next, one can check that

$$\langle \psi_{min}, \gamma^n \gamma^i \widehat{\gamma}^n \widehat{\gamma}^j \psi_{min} \rangle = \delta^{ij} 1_C + \langle \phi, (E^i - I^i)(E^j + I^j)\phi \rangle. \tag{2.25}$$

Hence

$$\langle \psi_{min}, \gamma^n \gamma^i \widehat{\gamma}^n \widehat{\gamma}^j \widehat{A}_{ji} \psi_{min} \rangle = (\partial f)^* H_{\partial M} 1_C + \widehat{A}_{ji} \langle \phi, (E^i - I^i)(E^j + I^j)\phi \rangle. \tag{2.26}$$

This proves the proposition. $\qquad \square$

The first two terms on the right-hand side of (2.19) give the Brown–York energy. (We have taken a particular normalization of the Brown–York energy.) In the case when $N$ is a small perturbation of $M$, the function $\phi$ will also be small. Hence Proposition 6 shows that in this weak field limit, to leading order the quasilocal energy $\mathcal{E}_C$ is the Brown–York energy.

*Remark 3.* With reference to Proposition 6, in the asymptotically flat case the geometry on most of a large domain $N$ will be a small perturbation of the Euclidean metric. Furthermore, it's known that the Brown–York energy of large spheres approaches the ADM mass (Ref. 19 and references therein). This makes it plausible that under an appropriate exhaustion of an asymptotically flat manifold, the quasilocal mass will approach the ADM mass. That this is true in the rotationally symmetric case follows from Proposition 9.

## D. Conformal deformations

In this section we use the conformal covariance of the Dirac operator to say something about the quasilocal energy; cf. Ref. 6, Sec. 4.

Put $(N', g') = (N, e^{2\phi}g)$, where $\phi|_{\partial N}$ is locally constant and $\phi|_C = 0$. Let $f : N \to M$ be as before. Suppose that $\mathrm{Ker}(\mathcal{D}) = 0$. Then $\overline{1}_C = 1_C$. The pure Dirac operator $Dirac^{N'}$ is related to the pure Dirac operator $Dirac^N$ by $Dirac^{N'} = e^{-\frac{n+1}{2}\phi} Dirac^N e^{\frac{n-1}{2}\phi}$ (Ref. 20, Sec. II). Thinking of $S_N \otimes f^* S_M^*$ as $\mathrm{Hom}(f^* S_M, S_N)$, the same argument gives $D^{N'} = e^{-\frac{n+1}{2}\phi} D^N e^{\frac{n-1}{2}\phi}$.

Let $\psi$ be a minimizer for the quasilocal energy of $N$ relative to $M$, with $\pi_+(\psi|_{\partial N}) = 1_C$. Putting $\psi' = e^{-\frac{n-1}{2}\phi}\psi$, it satisfies $D^{N'}\psi' = 0$ with $\pi_+(\psi'|_{\partial N'}) = 1_C$. As $\mathrm{Ker}(\mathcal{D}') = 0$, it follows that

$$\mathcal{E}_C' = -\int_{\partial N'} \langle \psi', \nabla_{e_n'}^{N'} \psi' \rangle \mathrm{dvol}_{g'}. \tag{2.27}$$

From Ref. 20, Sec. II,

$$\nabla_{e_n'}^{N'} = e^{-\phi}\left(\nabla_{e_n}^N + \frac{1}{4}(e_\alpha\phi)[\gamma^n, \gamma^\alpha]\right). \tag{2.28}$$

Applying this to (2.27), and using the fact that $\phi|_{\partial N}$ is locally constant, one finds

$$\mathcal{E}_C' - \mathcal{E}_C = \frac{n-1}{2}\int_{\partial N} (e_n\phi)|\psi|^2 \, \mathrm{dvol}_{\partial N}. \tag{2.29}$$

We now specialize to the case when $N = M$, $f = \mathrm{Id}$ and $C = \partial N$. We no longer have $\mathrm{Ker}(\mathcal{D}) = 0$.

*Lemma 4.* If $e_n\phi \geq 0$ then $\mathcal{E}_{\partial N'}' = \frac{n-1}{2}\int_{\partial N} (e_n\phi) \, \mathrm{dvol}_{\partial N}$. If $e_n\phi < 0$ then $\mathcal{E}_{\partial N'}' = -\infty$.

*Proof.* Note that $\phi|_{\partial N} = 0$. Given $\psi' \in C^\infty(N'; E')$ satisfying $D^{N'}\psi' = 0$ and $\pi_+(\psi'|_{\partial N}) = \overline{1}_{\partial N}'$, put $\psi = e^{\frac{n-1}{2}\phi}\psi'$. Then $D^N\psi = 0$ and $\pi_+(\psi|_{\partial N}) = \overline{1}_{\partial N}$. The Proof of Proposition 3 implies that $\psi = 1 + \rho$ for some harmonic form $\rho$ satisfying relative boundary conditions. As in the Proof of Proposition 3, we know that $\int_{\partial N} \langle \psi, \nabla_{e_n}^N \psi \rangle \mathrm{dvol}_g = 0$. Since

$$-\int_{\partial N'} \langle \psi', \nabla_{e_n'}^{N'}\psi' \rangle \mathrm{dvol}_{g'} + \int_{\partial N} \langle \psi, \nabla_{e_n}^N\psi \rangle \mathrm{dvol}_g = \frac{n-1}{2}\int_{\partial N} (e_n\phi)|\psi|^2 \, \mathrm{dvol}_{\partial N}, \tag{2.30}$$

if $e_n\phi \geq 0$ then we minimize $\frac{n-1}{2}\int_{\partial N} (e_n\phi)|\psi|^2 \, \mathrm{dvol}_{\partial N}$ by taking $\psi = 1$. As $\mathrm{dvol}_N \in \mathrm{Ker}(\mathcal{D})$, if $e_n\phi < 0$ then we can make $\frac{n-1}{2}\int_{\partial N} (e_n\phi)|\psi|^2 \, \mathrm{dvol}_{\partial N}$ arbitrarily negative by taking $\psi = 1 + s\,\mathrm{dvol}_N$ with $s$ large. $\square$

One can check that the mean curvatures of $\partial N'$ and $\partial N$ are related by

$$H_{\partial N'} = H_{\partial N} - (n-1)e_n\phi. \tag{2.31}$$

Hence if $e_n\phi \geq 0$ then

$$\mathcal{E}_{\partial N'}' = \frac{1}{2}\int_{\partial N} (H_{\partial N} - H_{\partial N'}) \, \mathrm{dvol}_{\partial N} = \frac{1}{2}\int_{\partial N'} (H_{\partial N} - H_{\partial N'}) \, \mathrm{dvol}_{\partial N'}, \tag{2.32}$$

showing that the quasilocal energy equals the Brown–York energy.

In this conformal setting, we can also express the quasilocal energy as an interior integral. If $n > 2$ then

$$\frac{n-1}{2}\int_{\partial N} (e_n\phi) \, \mathrm{dvol}_{\partial N} = \frac{n-1}{n-2}\int_{\partial N} \left(e_n e^{\frac{(n-2)\phi}{2}}\right) \mathrm{dvol}_{\partial N} \tag{2.33}$$

$$= -\frac{n-1}{n-2}\int_N \triangle\, e^{\frac{(n-2)\phi}{2}} \, \mathrm{dvol}_N$$

$$= \frac{1}{4}\int_N e^{\left(\frac{n}{2}+1\right)\phi}\left(R_{g'} - e^{-2\phi}R_g\right)\mathrm{dvol}_N.$$

Hence if $e_n \phi \geq 0$ then

$$\mathcal{E}'_{\partial N'} = \frac{1}{4} \int_N e^{\left(\frac{n}{2}+1\right)\phi} \left( R_{g'} - e^{-2\phi} R_g \right) \mathrm{dvol}_N \tag{2.34}$$

One can check that this is also true when $n = 2$.

As $|\Lambda^2(d\mathrm{Id})| = e^{-2\phi}$, for $\mathrm{Id} : (N', g') \to (N, g)$, Eq. (2.34) is consistent with Proposition 4.

As a special case, let $(M, g_{Eucl})$ be a compact connected codimension-zero submanifold of $\mathbb{R}^2$ with nonempty boundary. If $\phi \in C^\infty(M)$ vanishes on $\partial M$ and satisfies $e_n \phi \geq 0$ there, put $g' = e^{2\phi} g_{Eucl}$. Then the quasilocal energy of $(M, g')$, relative to $(M, g_{Eucl})$, is

$$\mathcal{E}_{\partial M} = \frac{1}{4} \int_M e^{2\phi} R_{g'} \, \mathrm{dvol}_{g_{Eucl}} = \frac{1}{4} \int_M R_{g'} \, \mathrm{dvol}_{g'}. \tag{2.35}$$

## E. Background space in $\mathbb{R}^n$

In this section we make the quasilocal energy more explicit when the background space $M$ is a domain in $\mathbb{R}^n$. We also treat the case of rotationally symmetric $N$.

From Corollary 5, if $M \subset \mathbb{R}^n$ and $R_N \geq 0$ then $\mathcal{E}_C \geq 0$.

*Proposition 7.* *If $M \subset \mathbb{R}^n$ then two choices of $f : N \to M$ with the same boundary restriction $\partial f$ will give the same value for $\mathcal{E}_C$.*

*Proof.* This is because the construction of $\mathcal{E}_C$ only involves $f$ through the boundary condition on $\partial f$ and the pullback of the connection on $S_M^*$. Since $S_M^*$ is a trivial bundle with trivial connection, the pullbacks of $S_M^*$ under two choices of $f$ will be equivalent. □

Of course, $\mathcal{E}_C$ still depends on the intrinsic geometry of $\partial N$, the extrinsic geometry of $\partial N$ (resp. $\partial M$) in $N$ (resp. $M$), and *a priori* the interior geometry of $N$.

*Proposition 8.* *Whenever $\mathrm{Ker}(\mathcal{D}) = 0$, the quasilocal energy can be described as follows. Within the spinor module associated to $\mathbb{R}^n$, let $\{\epsilon_a\}_{a=1}^{2^{\frac{n}{2}-1}}$ be an orthonormal basis for $\mathrm{Ker}(\gamma^0 \gamma^n - I)$ and let $\{\epsilon'_a\}_{a=1}^{2^{\frac{n}{2}-1}}$ be an orthonormal basis for $\mathrm{Ker}(\gamma^0 \gamma^n + I)$. Extending $\epsilon_a$ and $\epsilon'_a$ to constant-valued sections of $S_M$, let $\psi_a \in C^\infty(N; S_N)$ be a harmonic spinor on $N$ with boundary value in $(\partial f) * \epsilon_a \cdot 1_C + \mathrm{Ker}(\gamma^0 \gamma^n + I)$ and let $\psi'_a \in C^\infty(N; S_N)$ be a harmonic spinor on $N$ with boundary value in $(\partial f)^* \epsilon'_a \cdot 1_C + \mathrm{Ker}(\gamma^0 \gamma^n - I)$ Then*

$$\mathcal{E}_C = -2^{-n/2} \sum_{a=1}^{2^{\frac{n}{2}-1}} \int_{\partial N} \left( \langle \psi_a, \nabla^N_{e_n} \psi_a \rangle + \langle \psi'_a, \nabla^N_{e_n} \psi'_a \rangle \right) \mathrm{dvol}_{\partial N}. \tag{2.36}$$

*In particular, this is true under the hypotheses of Proposition 2.*

*Proof.* Let $\epsilon_a^* \in S_M^*$ denote inner product with $\epsilon_a$, and similarly for $\epsilon_a'^*$. Under the isomorphism $S_M \otimes S_M^* \cong \Lambda^* T^* M$, we claim that $\sum_a \left( \epsilon_a \otimes \epsilon_a^* + \epsilon'_a \otimes \epsilon_a'^* \right)$ corresponds to $2^{n/2} \in \Lambda^* T^* M$. This is because if $I$ is a nontrivial increasing multi-index with entries from $\{1, \ldots, n\}$ then $\mathrm{Tr}\left( \gamma^I \frac{\mathrm{Id}-\gamma^0\gamma^n}{2} \right) + \mathrm{Tr}\left( \gamma^I \frac{\mathrm{Id}+\gamma^0\gamma^n}{2} \right) = 0$, while $\mathrm{Tr}\left( \frac{\mathrm{Id}-\gamma^0\gamma^n}{2} \right) + \mathrm{Tr}\left( \frac{\mathrm{Id}+\gamma^0\gamma^n}{2} \right) = 2^{n/2}$.

On the other hand, if $\eta_a \in \mathrm{Ker}(\gamma^0 \gamma^n + I)$ then $T(\eta_a \otimes \epsilon_a^*) = -\eta_a \otimes \epsilon_a^*$, and similarly for an element of the form $\eta'_a \otimes \epsilon_a'^*$ with $\eta'_a \in \mathrm{Ker}(\gamma^0 \gamma^n + I)$. Thus

$$\Psi = 2^{-n/2} \sum_a \left( \psi_a \otimes f^* \epsilon_a^* + \psi'_a \otimes f^* \epsilon_a'^* \right) \tag{2.37}$$

satisfies $D^N \Psi = 0$ and $\pi_+ \left( \Psi|_{\partial N} \right) = 1_C$. As $\mathrm{Ker}(\mathcal{D}) = 0$, it is the unique such solution. Hence

$$\mathcal{E}_C = - \int_{\partial N} \langle \Psi, \nabla^N_{e_n} \Psi \rangle \, \mathrm{dvol}_{\partial N} \tag{2.38}$$

$$= -2^{-n/2} \sum_{a=1}^{2^{\frac{n}{2}-1}} \left( \int_{\partial N} \langle \psi_a, \nabla^N_{e_n} \psi_a \rangle \mathrm{dvol}_{\partial N} + \langle \psi'_a, \nabla^N_{e_n} \psi'_a \rangle \mathrm{dvol}_{\partial N} \right).$$

This proves the proposition. □

We now consider the case when $N$ is diffeomorphic to a disk, with a rotationally symmetric metric $g_N$. Then $(N, g_N)$ is conformally equivalent to a disk in $\mathbb{R}^n$ and by rescaling the disk, we can assume that the conformal equivalence $f$ is an isometry on the boundary.

*Proposition 9.* *In the rotationally symmetric case, if $H_{\partial N} > 0$ and the sectional curvature of $\partial N$ is $\frac{1}{k^2}$ then*

$$\mathcal{E}_{\partial N} = \frac{1}{2} \int_{\partial N} \left( \frac{n-1}{k} - H_{\partial N} \right) \mathrm{dvol}_{\partial N}. \tag{2.39}$$

*Proof.* This follows from (2.32). □

## F. Odd dimensional spaces

We now assume that $n$ is odd. The Clifford algebra has a faithful representation $S$ of complex dimension $2^{\frac{n+1}{2}}$, which breaks up into two isomorphic spinor representations $S^+ \oplus S^-$ of $\mathrm{Spin}(n)$. Explicitly, we can write Clifford generators $\{\gamma^\alpha\}_{\alpha=1}^n$ on $S$ as $\gamma^\alpha = \begin{pmatrix} 0 & \sigma^\alpha \\ \sigma^\alpha & 0 \end{pmatrix}$, where $\{\sigma^\alpha\}_{\alpha=1}^n$ satisfy the Clifford relations on $\mathbb{C}^{2^{\frac{n-1}{2}}}$. We again let $\epsilon$ be the $\mathbb{Z}_2$-grading operator on $S$.

With this definition of the spinors $S$, put $E = S_N \otimes f^* S_M^*$. We can identify $E|_{\partial N} \cong \mathrm{End}(S_N|_{\partial N})$ with $\Lambda^*(T^*N)|_{\partial N} \oplus \Lambda^*(T^*N)|_{\partial N}$. To realize this identification explicitly, put $\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $I$ be an increasing multi-index with entries between $1$ and $n$. Let $\tau^0$ denote a new odd variable. Then an element $\omega_I \tau^I + \tau^0 \wedge \omega_I' \tau^I$ acts on $S_N|_{\partial N}$ by sending $\psi$ to $(\omega_I \gamma^I + \gamma^0 \omega_I' \gamma^I)\psi$.

Define $T$ as in (2.8). For $\omega_1, \omega_2, \omega_1', \omega_2' \in \Lambda^* T^* \partial N$, we have

$$\big((\omega_1 + \tau^n \wedge \omega_2) + \tau^0 \wedge (\omega_1' + \tau^n \wedge \omega_2')\big)\gamma^0 \gamma^n \psi \qquad (2.40)$$
$$= \gamma^0 \gamma^n \big((\omega_1 - \tau^n \wedge \omega_2) + \tau^0 \wedge (-\omega_1' + \tau^n \wedge \omega_2')\big)\psi.$$

Hence the induced action of $T$ on $E|_{\partial N} \cong \mathrm{End}(S_N)|_{\partial N}$ is

$$T\big(\omega_1 + \tau^n \wedge \omega_2, \omega_1' + \tau^n \wedge \omega_2'\big) = (\omega_1 - \tau^n \wedge \omega_2, -\omega_1' + \tau^n \wedge \omega_2'). \qquad (2.41)$$

We consider solutions to the Dirac equation $D^N \psi = 0$ on sections $\psi \in C^\infty(N; E)$ with the boundary condition that if $\psi|_{\partial N} = \big(\omega_1 + \tau^n \wedge \omega_2, \omega_1' + \tau^n \wedge \omega_2'\big)$ then $\omega_1 = \overline{1}_C$ and $\omega_2 = \omega_1' = \omega_2' = 0$, where the overline denotes an orthogonal projection as before. Then the results of the previous sections have straightforward extensions.

## III. LORENTZIAN CASE

This section is about the extension of Sec. II to hypersurfaces in Lorentzian manifolds. Section III A has background material. Section III B discusses the case when the background space $M$ is a totally geodesic hypersurface in a Lorentzian manifold $\overline{M}$. Section III C deals with the case when $M$ is a compact spatial hypersurface-with-boundary in $\mathbb{R}^{n,1}$. Finally, in Sec. III D we describe how the results of Sec. III C extend to when $M$ is a compact spatial hypersurface-with-boundary in a product spacetime $\mathbb{R} \times X$.

### A. Background information

Let $\overline{N}$ and $\overline{M}$ be $(n+1)$-dimensional Lorentzian manifolds with signature $(-1, 1, \ldots, 1)$. Let $N$ and $M$ be compact connected $n$-dimensional spacelike submanifolds of $\overline{N}$ and $\overline{M}$, respectively, with nonempty boundary. Let $f : N \to M$ be a smooth spin map so that for each connected component $Z$ of $\partial N$, the map $\partial f$ restricts to an isometric diffeomorphism from $Z$ to $(\partial f)(Z)$. By shrinking $\overline{N}$ and $\overline{M}$ to suitable neighborhoods of $N$ and $M$, respectively, we can assume that $f$ is the restriction of a spin map $\overline{f} : \overline{N} \to \overline{M}$. For simplicity, we will assume that $\overline{N}$ and $\overline{M}$ are spin and that $\partial f$ is a spin diffeomorphism on components of $\partial N$. We let $S_{\overline{N}}$ be the standard spinor bundle on $\overline{N}$, and similarly for $S_{\overline{M}}$.

We can identify $S_{\overline{N}}|_N$ with $S_N$, and similarly for $S_{\overline{M}}$.

*Remark 4.* We could phrase what follows just in terms of $N$, $M$ and their normal bundles, but it seems more illuminating to include the ambient spaces $\overline{N}$ and $\overline{M}$.

*Remark 5.* To clarify the relation between the spinor bundle $S_N$ considered here and that considered in Sec. II, we mention some facts about spinors. Suppose first that $n$ is odd. Then $\dim(S_{\mathbb{R}^{n,1}}) = 2^{\frac{n+1}{2}}$ and $\dim(S_{\mathbb{R}^{n,1}} \otimes S_{\mathbb{R}^{n,1}}^*) = 2^{n+1}$, which equals $\dim(\Lambda^* \mathbb{R}^{n,1})$. We can identify $S_{\mathbb{R}^{n,1}} \otimes S_{\mathbb{R}^{n,1}}^*$ with $\Lambda^* \mathbb{R}^{n,1}$. Using a timelike unit vector $e_0$, the latter can be identified with $\Lambda^* \mathbb{R}^n \widehat{\otimes} \Lambda^* \mathbb{R}^1$. Compare with Sec. II E.

Now suppose that $n$ is even. Then $\dim(S_{\mathbb{R}^{n,1}}) = 2^{\frac{n}{2}}$ and $\dim(S_{\mathbb{R}^{n,1}} \otimes S_{\mathbb{R}^{n,1}}^*) = 2^n$. On the other hand, $\dim(\Lambda^* \mathbb{R}^{n,1}) = 2^{n+1}$. It turns out that one can identify $S_{\mathbb{R}^{n,1}} \otimes S_{\mathbb{R}^{n,1}}^*$, as a $\mathrm{Spin}(n,1)$-module, with $\Lambda^*(\mathbb{R}^{n,1})/(\omega \sim *\omega)$. Using a timelike unit vector $e_0$, the latter can be identified with $\Lambda^* \mathbb{R}^n$. Compare with Sec. II A.

The restriction to $N$ of the connection on $S_{\overline{N}}$ has the local form

$$\nabla_\sigma^W = e_\sigma + \frac{1}{8}\omega_{\alpha\beta\sigma}[\gamma^\alpha, \gamma^\beta] + \frac{1}{2}\omega_{0\alpha\sigma}\gamma^0\gamma^\alpha, \qquad (3.1)$$

where $\alpha$ and $\beta$ are summed from 1 to $n$. Note that $\nabla^W$ is generally not a unitary connection, because of the last term on the right-hand side of (3.1). However, the ensuing Dirac-type operator

$$D^W = -\sqrt{-1} \sum_{\sigma=1}^{n} \gamma^\sigma \nabla_\sigma^W \tag{3.2}$$

is formally self-adjoint. To verify this, one can use the fact that in normal coordinates around a point of $N$, one has

$$[\nabla_\sigma^W, \gamma^\alpha] = \omega_{0\alpha\sigma} \gamma^0. \tag{3.3}$$

The restriction of $S_{\overline{M}}$ to $M$ is isomorphic to $S_M$. There are corresponding connections on $S_M^*$. As in Sec. II, it will be convenient to use the isomorphism of $\mathrm{Spin}(n,1)$-modules $S_M^* \cong S_M$ to transfer the connection from $S_M^*$ to $S_M$. There is a subtlety in that the isomorphism is not unitary. If $\sigma \in \mathrm{Spin}(n,1)$ is a transition function for $S_M$ then the corresponding transition function for $S_M^*$ is $\sigma^{-T}$. This is related to $\sigma$ by $\sigma^{-T} = C\sigma C^{-1}$, where $C$ is the charge conjugation matrix. In short, when written on $S_M$, one finds that the connection on $S_M^*$ takes the local form

$$\widehat{\nabla}_\sigma^W = e_{\widehat{\sigma}} + \frac{1}{8} \widehat{\omega}_{\widehat{\alpha}\widehat{\beta}\sigma} [\widehat{\gamma}^{\widehat{\alpha}}, \widehat{\gamma}^{\widehat{\beta}}] - \frac{1}{2} \widehat{\omega}_{\widehat{0}\widehat{\alpha}\sigma} \widehat{\gamma}^{\widehat{0}} \widehat{\gamma}^{\widehat{\alpha}}. \tag{3.4}$$

Note the change in sign in the last term as compared with (3.1).

Let $\nabla^N$ be the connection on $E = S_N \otimes f^* S_M^*$. It takes the local form

$$\nabla_\sigma^N = e_\sigma + \frac{1}{8} \omega_{\alpha\beta\sigma} [\gamma^\alpha, \gamma^\beta] + \frac{1}{2} \omega_{0\alpha\sigma} \gamma^0 \gamma^\alpha + \frac{1}{8} \widehat{\omega}_{\widehat{\alpha}\widehat{\beta}\sigma} [\widehat{\gamma}^{\widehat{\alpha}}, \widehat{\gamma}^{\widehat{\beta}}] - \frac{1}{2} \widehat{\omega}_{\widehat{0}\widehat{\alpha}\sigma} \widehat{\gamma}^{\widehat{0}} \widehat{\gamma}^{\widehat{\alpha}}. \tag{3.5}$$

The corresponding Dirac-type operator

$$D^N = -\sqrt{-1} \sum_{\sigma=1}^{n} \gamma^\sigma \nabla_\sigma^N \tag{3.6}$$

is formally self-adjoint except for the term $\sqrt{-1}\gamma^\sigma \cdot \frac{1}{2} \widehat{\omega}_{\widehat{0}\widehat{\alpha}\sigma} \widehat{\gamma}^{\widehat{0}} \widehat{\gamma}^{\widehat{\alpha}}$.

## B. Time-symmetric background space

Suppose that $M$ is a totally geodesic subspace of $\overline{M}$. Then $\widehat{\omega}_{\widehat{0}\widehat{\alpha}\sigma} = 0$. We are in a situation analogous to Sec. II, except that $N$ is now a hypersurface-with-boundary in a Lorentzian manifold. We give the extensions of results from Sec. II.

Let $C$ be a nonempty union of connected components of $\partial N$. We define the quasilocal energy $\mathcal{E}_C$ as in Definition 2.

*Lemma 5.* If $f : N \to M$ is an isometric diffeomorphism, and $N$ is totally geodesic in $\overline{N}$, then $\mathcal{E}_C = 0$.

*Proof.* The proof is the same as that of Proposition 3. $\square$

Define $T_{AB} = R_{AB}^{\overline{N}} - \frac{1}{2} R_{\overline{N}} g_{AB}^{\overline{N}}$, for $0 \le A, B \le n$.

*Proposition 10.* Suppose that $M$ has nonnegative curvature operator. If

$$2\left(T_{00} - \sqrt{-\sum_{\alpha=1}^{n} T_{0\alpha} T^{0\alpha}}\right) \ge |\Lambda^2 df|(f^* R_M) \tag{3.7}$$

then $\mathcal{E}_C \ge 0$.

*Proof.* Using the calculations in Ref. 2, the analog of (2.3) is

$$(D^N)^2 = (\nabla^N)^* \nabla^N + \frac{1}{2}\left(T_{00} + T_{0\alpha}\gamma^0\gamma^\alpha\right) - \frac{1}{4}[\gamma^\sigma, \gamma^\tau]\left(\frac{1}{8}\widehat{R}_{\widehat{\alpha}\widehat{\beta}\sigma\tau}[\widehat{\gamma}^{\widehat{\alpha}}, \widehat{\gamma}^{\widehat{\beta}}]\right). \tag{3.8}$$

If $D^N \psi = 0$, we obtain

$$-\int_{\partial N} \langle \psi, \nabla_{e_n}^E \psi \rangle \, \mathrm{dvol} = \int_N |\nabla^N \psi|^2 \, \mathrm{dvol}$$
$$+ \int_N \langle \psi, \left(\frac{1}{2}\left(T_{00} + T_{0\alpha}\gamma^0\gamma^\alpha\right) - \frac{1}{4}[\gamma^\sigma, \gamma^\tau]\left(\frac{1}{8}\widehat{R}_{\widehat{\alpha}\widehat{\beta}\sigma\tau}[\widehat{\gamma}^{\widehat{\alpha}}, \widehat{\gamma}^{\widehat{\beta}}]\right)\right)\psi \rangle \, \mathrm{dvol}. \tag{3.9}$$

Using (2.7), the proposition follows. $\square$

We now give an analog of (2.19). If $D^N \psi = 0$ then the analog of (2.5) is

$$\nabla^N_{e_n} \psi = D^{\partial N}_{Riem} \psi + \frac{H_{\partial N}}{2} - \frac{1}{2} \gamma^n \gamma^i \widehat{\gamma}^{\widehat{n}} \widehat{\gamma}^{\widehat{j}} \widehat{A_{\widehat{j}i}} \psi - \frac{1}{2} \omega_{0ii} \gamma^0 \gamma^n \psi + \frac{1}{2} \omega_{0ni} \gamma^0 \gamma^i \psi, \tag{3.10}$$

where $D^{\partial N}_{Riem}$ is the intrinsic Riemannian Dirac-type operator on $\partial N$. Then

$$- \int_{\partial N} \langle \psi, \nabla^E_{e_n} \psi \rangle \, \mathrm{dvol} = - \int_{\partial N} \langle \psi, D^{\partial N}_{Riem} \psi \rangle \, \mathrm{dvol}_{\partial N} - \frac{1}{2} \int_{\partial N} H_{\partial N} |\psi|^2 \, \mathrm{dvol}_{\partial N} \tag{3.11}$$
$$+ \frac{1}{2} \int_{\partial N} \langle \psi, \gamma^n \gamma^i \widehat{\gamma}^{\widehat{n}} \widehat{\gamma}^{\widehat{j}} \widehat{A_{\widehat{j}i}} \psi \rangle \, \mathrm{dvol}_{\partial N}$$
$$+ \frac{1}{2} \int_{\partial N} \langle \psi, \gamma^0 \gamma^n \omega_{0ii} \psi \rangle \, \mathrm{dvol}_{\partial N}$$
$$- \frac{1}{2} \int_{\partial N} \langle \psi, \gamma^0 \gamma^i \omega_{0ni} \psi \rangle \, \mathrm{dvol}_{\partial N}.$$

If $\psi \in \mathcal{C}_C$ then the last term in (3.11) vanishes, as $T$ anticommutes with $\gamma^0 \gamma^i$,

To analyze the next-to-last term in (3.11), suppose that $n$ is even and $\overline{1}_C = 1_C$. Writing $\psi_{min}|_{\partial N} = 1_C + \tau^n \wedge \phi$ for $\phi \in \Omega^*(\partial N)$, we have $\gamma^0 \gamma^n \psi_{min}|_{\partial N} = -\gamma^n \gamma^0 (1_C + \tau^n \wedge \phi)$. The $\mathbb{Z}_2$-grading operator $\epsilon$ of $S_N$, when acting on $S_N \otimes S_N^* \cong \Lambda^* T^* N$, is a fourth root of 1 (depending on the congruence class of $n$ modulo 4) times

$$\gamma^1 \gamma^2 \ldots \gamma^n = (-1)^{\frac{n}{2}} (E^1 - I^1)(E^2 - I^2) \ldots (E^n - I^n). \tag{3.12}$$

Applying both sides to $1_C$ gives $\gamma^0 1_C = c 1_C \tau^1 \wedge \ldots \wedge \tau^n$ for a complex constant $c$ of unit norm, depending on $n$. Then $\gamma^n \gamma^0 1_C = \sqrt{-1}$ $(E^n - I^n) \cdot c 1_C (\tau^1 \wedge \ldots \wedge \tau^n) = c' 1_C \tau^1 \wedge \ldots \wedge \tau^{n-1}$, for another unit constant $c'$. In particular, the following terms vanish:

$$\langle 1_C, \gamma^0 \gamma^n \omega_{0ii} 1_C \rangle = \langle \tau^n \wedge \phi, \gamma^0 \gamma^n \omega_{0ii} 1_C \rangle = \langle 1_C, \gamma^0 \gamma^n \omega_{0ii} \tau^n \wedge \phi \rangle = 0. \tag{3.13}$$

As in (2.19), we obtain

$$\mathcal{E}_C = -\frac{1}{2} \int_C H_{\partial N} \, \mathrm{dvol}_{\partial N} + \frac{1}{2} \int_C (\partial f)^* H_{\partial M} \, \mathrm{dvol}_{\partial N} + Q(\phi), \tag{3.14}$$

where $Q(\phi)$ is an explicit homogeneous expression of order two in $\phi$. There is a similar discussion when $n$ is odd.

If $K$ denotes the second fundamental form of $N$ in $\overline{N}$ then the sum $\omega_{0ii}$ on $\partial N$, which will appear in the next proposition, equals $\mathrm{Tr}(K) - K(e_n, e_n)$.

**Proposition 11.** *Suppose that*

- *$M$ is a convex domain in $\mathbb{R}^n$,*
- *$T_{00} - \sqrt{-\sum_{\alpha=1}^n T_{0\alpha} T^{0\alpha}} \geq 0$,*
- *$H_{\partial N} - |\omega_{0ii}| \geq (\partial f) * H_{\partial M}$, and*
- *$T_{00} - \sqrt{-\sum_{\alpha=1}^n T_{0\alpha} T^{0\alpha}} > 0$ somewhere or $H_{\partial N} - |\omega_{0ii}| > (\partial f) * H_{\partial M}$ somewhere.*
  *Then $\mathrm{Ker}(\mathcal{D}) = 0$.*

*Proof.* The proof is similar to that of Proposition 2. $\square$

**Proposition 12.** *Whenever $\mathrm{Ker}(\mathcal{D}) = 0$, the quasilocal energy can be described as follows. Within the spinor module associated to $\mathbb{R}^n$, let $\{\epsilon_a\}_{a=1}^{2^{\frac{n}{2}-1}}$ be an orthonormal basis for $\mathrm{Ker}(\gamma^0 \gamma^n - I)$ and let $\{\epsilon'_a\}_{a=1}^{2^{\frac{n}{2}-1}}$ be an orthonormal basis for $\mathrm{Ker}(\gamma^0 \gamma^n + I)$. Extending $\epsilon_a$ and $\epsilon'_a$ to constant-valued sections of $S_M$, let $\psi_a \in C^\infty(N; S_N)$ be a harmonic spinor on $N$ with boundary value in $(\partial f) * \epsilon_a \cdot 1_C + \mathrm{Ker}(\gamma^0 \gamma^n + I)$ and let $\psi'_a \in C^\infty(NS_N)$ be a harmonic spinor on $N$ with boundary value in $(\partial f)^* \epsilon'_a \cdot 1_C + \mathrm{Ker}(\gamma^0 \gamma^n - I)$ Then*

$$\mathcal{E}_C = -2^{-n/2} \sum_{a=1}^{2^{\frac{n}{2}-1}} \int_{\partial N} \left( \langle \psi_a, \nabla^N_{e_n} \psi_a \rangle + \langle \psi'_a, \nabla^N_{e_n} \psi'_a \rangle \right) \mathrm{dvol}_{\partial N}. \tag{3.15}$$

*In particular, this is true under the hypotheses of Proposition 11.*

*Proof.* The proof is the same as for Proposition 8. $\square$

## C. Background space in $\mathbb{R}^{n,1}$

We now let the background space be a more general submanifold of Minkowski space. Let $\overline{M}$ be the Lorentzian $\mathbb{R}^{n,1}$ and suppose that $M$ is a compact connected spacelike hypersurface-with-boundary in $\overline{M}$. In general, the term $\widehat{\omega}_{\widehat{0}\widehat{\alpha}\sigma}$ in (3.5) need not be zero and there is an apparent problem with formal self-adjointness of $D^N$ on the interior of $N$. We can get around this by changing the inner product. We use the fact that $S_M$ is the restriction of the trivial bundle $S_{\overline{M}}$ to $M$. Choose a constant timelike unit vector field $\mathcal{T}$ in $\overline{M}$. There is a corresponding inner product $\langle\cdot,\cdot\rangle_{\mathcal{T}}$ on the trivial bundle $S_{\overline{M}}$. Namely, if $(\cdot,\cdot)$ is the indefinite $\mathrm{Spin}(n,1)$-invariant bilinear form on $S_{\overline{M}}$ then the inner product on $S_{\overline{M}}$ is given by $\langle s_1, s_2\rangle_{\mathcal{T}} = (\sqrt{-1}c(\mathcal{T})s_1, s_2)$, where $c(\mathcal{T})$ is Clifford multiplication by $\mathcal{T}$. We can then restrict $(S_{\overline{M}}, \langle\cdot,\cdot\rangle_{\mathcal{T}})$ to $M$, dualize to get an inner product $\langle\cdot,\cdot\rangle_{sa}$ on $S_M^*$ and pullback to $N$, to obtain an inner product $\langle\cdot,\cdot\rangle_{E,sa}$ on $E = S_N \otimes f^* S_M^*$.

In order to express the boundary conditions, it will be convenient to write things more intrinsically on $M$. Let $U \subset M$ be a connected open subset on which an oriented orthonormal frame $\{\widehat{e}_\alpha\}_{\widehat{\alpha}=1}^n$ is defined and let $m_0 \in U$ be a basepoint. Since the connection on $S_M^*$ is flat, we can locally write the expression $\frac{1}{8}\widehat{\omega}_{\widehat{\alpha}\widehat{\beta}\sigma}[\widehat{\gamma}^{\widehat{\alpha}}, \widehat{\gamma}^{\widehat{\beta}}] \otimes \widehat{\tau}^{\widehat{\sigma}} - \frac{1}{2}\widehat{\omega}_{\widehat{0}\widehat{\alpha}\sigma}\widehat{\gamma}^{\widehat{0}}\widehat{\gamma}^{\widehat{\alpha}} \otimes \widehat{\tau}^{\widehat{\sigma}}$ in (3.4) as $g^{-1}dg$ for some $g : U \to \mathrm{Spin}(n,1)$. Let $\rho : \mathrm{Spin}(n,1) \to \mathrm{SO}(n,1)^+$ be the double cover. If $\widehat{e_0}(m_0)$ is a unit normal vector to $M$ at $m_0$ then we can partially normalize $g$ by specifying that $g(m_0)$ sends $\widehat{e_0}(m_0)$ to $\mathcal{T}$ in $T_{m_0}\overline{M}$. The remaining ambiguity in $g$ will be left multiplication by a constant element of $\mathrm{Spin}(n)$. On the other hand, a change of oriented orthonormal frame corresponds to a map $\phi : U \to \mathrm{SO}(n)$ which lifts, using the spin structure, to a a map $\widetilde{\phi} : U \to \mathrm{Spin}(n)$. The effect of the new frame is to multiply $g$ on the right by $\widetilde{\phi}^{-1}$.

If $U_1$ and $U_2$ are overlapping such domains then their corresponding orthonormal frames will be related by a map $\phi_{12} : U_1 \cap U_2 \to \mathrm{SO}(n)$, which we lift using the spin structure to $\widetilde{\phi}_{12} : U_1 \cap U_2 \to \mathrm{Spin}(n)$. Then the maps $g_i : U_i \to \mathrm{Spin}(n,1)$ are related on $U_1 \cap U_2$ by $g_1 = u_{12}g_2\widetilde{\phi}_{12}^{-1}$, for some constant element $u_{12} \in \mathrm{Spin}(n)$. In effect, this uses the fact that $\widehat{e_0}$ is well-defined on $M$. Similarly if $U_1 \cap U_2$ intersects $\partial M$ then using the fact that the unit normal $\widehat{e_n}$ is well-defined, we can assume that $\widetilde{\phi}_{12}|_{U_1 \cap U_2 \cap \partial M}$ takes values in $\mathrm{Spin}(n-1)$.

On the domain $U$, put $A = g^* g$ and define a weighted inner product $\langle\cdot,\cdot\rangle_{S_M^*,sa}$ by $\langle s_1, s_2\rangle_{S_M^*,sa} = \langle s_1, As_2\rangle_{S_M^*}$ for $s_1, s_2 \in S_M^*$. Left multiplication of $g$ by an element of $\mathrm{Spin}(n)$ doesn't change $A$, so $A$ is independent of choices. The connection (3.4) on $S_M^*$ is compatible with $\langle\cdot,\cdot\rangle_{S_M^*,sa}$, since the ambient flat connection on $S_{\overline{M}}$ is compatible with $\langle\cdot,\cdot\rangle_{\mathcal{T}}$.

Given a smooth spin map $f : (N, \partial N) \to (M, \partial M)$ which is an isometric spin diffeomorphism on each connected component of $\partial N$, we construct the Clifford module $E = S_N \otimes f^* S_M^*$ on $N$, with the inner product $\langle\cdot,\cdot\rangle_{E,sa} = \langle\cdot,\cdot\rangle|_{S_N} \otimes f^* \langle\cdot,\cdot\rangle|_{S_M^*,sa}$ The analog of (2.2) is

$$\int_N \langle D^N\psi_1, \psi_2\rangle_{E,sa} \, \mathrm{dvol}_N - \int_N \langle\psi_1, D^N\psi_2\rangle_{E,sa} \, \mathrm{dvol}_N \tag{3.16}$$
$$= -\sqrt{-1}\int_{\partial N} \langle\psi_1, \gamma^n\psi_2\rangle_{E,sa} \, \mathrm{dvol}_{\partial N}.$$

*Proposition 13.* If $N$ satisfies the dominant energy condition $T_{00} \geq \sqrt{-\sum_{\alpha=1}^n T_{0\alpha}T^{0\alpha}}$ then for any $\psi \in C^\infty(N; E)$ satisfying $D^N\psi = 0$, we have

$$-\int_{\partial N} \langle\psi, \nabla_{e_n}^N\psi\rangle_{E,sa} \, \mathrm{dvol}_{\partial N} \geq 0. \tag{3.17}$$

*Proof.* On the interior of $N$, we can think of $D^N$ as $D^W \otimes \mathrm{Id}_{\mathbb{C}^{2^{n/2}}}$. By the calculations in Ref. 2, in this representation we can write

$$(D^N)^2 = \left((\nabla^N)_{sa}^* \nabla^N + \frac{1}{2}(T_{00} + T_{0\alpha}\gamma^0\gamma^\alpha)\right) \otimes \mathrm{Id}_{\mathbb{C}^{2^{n/2}}}. \tag{3.18}$$

As $D^N\psi = 0$, after integrating by parts we obtain from (3.18) that

$$-\int_{\partial N} \langle\psi, \nabla_{e_n}^N\psi\rangle_{E,sa} \, \mathrm{dvol}_{\partial N} \tag{3.19}$$
$$= \int_N \left(\langle\nabla\psi, \nabla\psi\rangle_{E,sa} + \frac{1}{2}\langle\psi, (T_{00} + T_{0\alpha}\gamma^0\gamma^\alpha)\psi\rangle_{E,sa}\right)\mathrm{dvol}.$$

The proposition follows. □

To follow what was done in Sec. II, we put $T = \gamma^0\gamma^n\widehat{\gamma}^{\widehat{0}}\widehat{\gamma}^{\widehat{n}}$, acting on $E|_{\partial N}$. We again have $T1 = 1$. Note that $T$ may not be self-adjoint with respect to $\langle\cdot,\cdot\rangle_{E,sa}$.

Given $\sigma \in \Omega^*(\partial N)$ with $T\sigma = \sigma$, there are two natural boundary conditions to impose on the equation $D^N\psi = 0$; we could ask that $\psi|_{\partial N} \in \sigma + \mathrm{Ker}(T+I)$ or we could ask that $\psi|_{\partial N} \in \sigma + (\mathrm{Ker}(T-I))^\perp$. For concreteness, we take the second choice.

Consider the operator $D^N$ acting on elements $\psi \in C^\infty(N; E)$ that satisfy $\psi|_{\partial N} \in (\mathrm{Ker}(T-I))^\perp$. This defines an elliptic boundary condition. Let $\mathcal{D}$ be the corresponding operator, densely defined on $C^\infty(N; E)$. Here $\mathcal{D}$ may not be self-adjoint but it still has discrete spectrum with eigenspaces of finite multiplicity.

We wish to find some $\psi \in C^\infty(N; E)$ so that $D^N \psi = 0$ and $\psi|_{\partial N} \in \sigma + (\mathrm{Ker}(T-I))^\perp$. This is an elliptic boundary value problem. Suppose that we can do this. Note that $\gamma^n \psi|_{\partial N} \in \gamma^n \sigma + (\mathrm{Ker}(T+I))^\perp$ Put

$$\mathcal{H} = \left\{ \eta \in C^\infty(N;E) \ : \ D^N \eta = 0, \eta|_{\partial N} \in \mathrm{Ker}(T+I) \right\}. \tag{3.20}$$

If $\eta \in \mathcal{H}$ then from (3.16), we see that $\int_{\partial N} \langle \sigma, \gamma^n \eta \rangle \mathrm{dvol}_{\partial N} = 0$. Conversely, given $\sigma \in \Omega^*(\partial N)$ with $T\sigma = \sigma$, Fredholm theory implies that if $\int_{\partial N} \langle \sigma, \gamma^n \eta \rangle \mathrm{dvol}_{\partial N} = 0$ for all $\eta \in \mathcal{H}$ then there is some $\psi \in C^\infty(N; E)$ so that $D^N \psi = 0$ and $\psi|_{\partial N} \in \sigma + (\mathrm{Ker}(T-I))^\perp$; cf. Ref. 21, Theorem 2.4.5. The section $\psi$ is unique up to addition by elements of $\mathrm{Ker}(\mathcal{D})$.

As before, $C$ is a nonempty union of connected components of $\partial N$. We would like to take $\sigma = 1_C$. To be sure that $\psi$ exists, we may have to slightly modify this choice. Let $P_{\gamma^n \mathcal{H}}$ be orthogonal projection from $\Omega^*(\partial N)$ to $\gamma^n \mathcal{H}$. We put $\bar{1}_C = 1_C - P_{\gamma^n \mathcal{H}} 1_C$, take $\sigma = \bar{1}_C$ and construct $\psi$ accordingly.

*Definition 3.  Put*

$$\mathcal{C}_C = \left\{ \psi \in C^\infty(N;E) \ : \ D^N \psi = 0, \psi|_{\partial N} \in \bar{1}_C + (\mathrm{Ker}(T-I))^\perp \right\}. \tag{3.21}$$

*The quasilocal energy is*

$$\mathcal{E}_C = \inf_{\psi \in \mathcal{C}} \left( -\int_{\partial N} \langle \psi, \nabla^N_{e_n} \psi \rangle_{E,sa} \, \mathrm{dvol}_{\partial N} \right). \tag{3.22}$$

*Proposition 14. If $\overline{N} = \overline{M} = \mathbb{R}^{n,1}$ and $N = M$ then $\mathcal{E}_{\partial N} = 0$.*

*Proof.* In this case, the connection forms satisfy $\widehat{\omega} = \omega$. Suppose that $n$ is odd. We can identify $E$ with $\Lambda^*(T^*N) + \tau^0 \wedge \Lambda^*(T^*N)$. Putting $\gamma^\alpha = \sqrt{-1}(E^\alpha - I^\alpha)$, $\widehat{\gamma^\alpha} = E^\alpha + I^\alpha$, $\gamma^0 = \sqrt{-1}(E^0 + I^0)$ and $\widehat{\gamma^0} = -(E^0 - I^0)$, when acting on $\Omega^*(N) \oplus \tau^0 \wedge \Omega^*(N)$ one can check that

$$\nabla^N_\sigma = e_\sigma + \omega_{\alpha\beta\sigma} E^\alpha I^\beta + \omega_{0\alpha\sigma}(E^0 I^\alpha + E^\alpha I^0). \tag{3.23}$$

In particular, $\nabla^N_\sigma 1 = 0$, so $D^N 1 = 0$. It follows that $\bar{1} = 1$, so $1 \in \mathcal{C}$, with $-\int_{\partial N} \langle 1, \nabla^N_{e_n} 1 \rangle_{E,sa} \, \mathrm{dvol}_{\partial N} = 0$. Proposition 13 now implies that $\mathcal{E}_{\partial N} = 0$. The proof when $n$ is even is similar. $\square$

For an alternative approach that maintains self-adjointness, if $U \subset M$ is an open set as before then on $U \cap \partial M$, we put

$$T_{sa} = \left[ (\partial f)^*(g^* g) \right]^{-\frac{1}{2}} \gamma^0 \gamma^n \widehat{\gamma^0} \widehat{\gamma^n} \left[ (\partial f)^*(g^* g) \right]^{\frac{1}{2}}, \tag{3.24}$$

which is now a self-adjoint idempotent with respect to $\langle \cdot, \cdot \rangle_{E,sa}$. To see that this is globally defined, as mentioned before on an overlap $U_1 \cap U_2 \cap \partial M$ we can assume that $\widetilde{\phi}_{12}$ commutes with $\widehat{\gamma^0}$ and $\widehat{\gamma^n}$. As

$$\left( (u_{12} g \widetilde{\phi}_{12}^{-1})^* u_{12} g \widetilde{\phi}_{12}^{-1} \right)^{\frac{1}{2}} = \widetilde{\phi}_{12}(g^* g)^{\frac{1}{2}} \widetilde{\phi}_{12}^{-1}, \tag{3.25}$$

we obtain

$$\left[ (\partial f)^*(g_1^* g_1) \right]^{-\frac{1}{2}} \gamma^0 \gamma^n \widehat{\gamma^0} \widehat{\gamma^n} \left[ (\partial f)^*(g_1^* g_1) \right]^{\frac{1}{2}}$$
$$= (\partial f)^* \widetilde{\phi}_{12} \left\{ \left[ (\partial f)^*(g_2^* g_2) \right]^{-\frac{1}{2}} \gamma^0 \gamma^n \widehat{\gamma^0} \widehat{\gamma^n} \left[ (\partial f)^*(g_2^* g_2) \right]^{\frac{1}{2}} \right\} (\partial f)^* \widetilde{\phi}_{12}^{-1}. \tag{3.26}$$

Taking into account that $\widetilde{\phi}_{12}$ encodes how spinors transform, it follows that $T_{sa}$ is globally defined on $\partial N$. We still have $T_{sa} \gamma^n + \gamma^n T_{sa} = 0$.

*Lemma 6. Consider the operator $D^N$ on elements $\psi \in C^\infty(N; E)$ satisfying the boundary condition*

$$(T_{sa} + I)\psi|_{\partial N} = 0. \tag{3.27}$$

*It is formally self-adjoint.*

*Proof.* If $\psi_1, \psi_2 \in C^\infty(N; E)$ satisfy $(T_{sa} + I)\psi_1|_{\partial N} = (T_{sa} + I)\psi_2|_{\partial N} = 0$ then

$$\int_{\partial N} \langle \psi_1, \gamma^n \psi_2 \rangle_{E,sa} \, \mathrm{dvol}_{\partial N} = -\int_{\partial N} \langle \psi_1, \gamma^n T_{sa}\psi_2 \rangle_{E,sa} \, \mathrm{dvol}_{\partial N} \tag{3.28}$$

$$= \int_{\partial N} \langle \psi_1, T_{sa}\gamma^n \psi_2 \rangle_{E,sa} \, \mathrm{dvol}_{\partial N}$$

$$= \int_{\partial N} \langle T_{sa}\psi_1, \gamma^n \psi_2 \rangle_{E,sa} \, \mathrm{dvol}_{\partial N}$$

$$= -\int_{\partial N} \langle \psi_1, \gamma^n \psi_2 \rangle_{E,sa} \, \mathrm{dvol}_{\partial N},$$

so $\int_{\partial N} \langle \psi_1, \gamma^n \psi_2 \rangle_{E,sa} \, \mathrm{dvol}_{\partial N}$ vanishes. Using (3.16), the lemma follows. $\square$

With reference to Lemma 6, let $\mathcal{D}$ be the self-adjoint extension. It has compact resolvent.

To find the right boundary value problem, we want a section $\sigma$ of $E|_{\partial N}$ so that $T_{sa}\sigma = \sigma$. Identifying $E|_{\partial N}$ with $\mathrm{End}(S_N|_{\partial N})$, an initial attempt would be the operator $s_C = 2^{-n/2} 1_C [(\partial f)^* (g^*g)]^{-\frac{1}{2}}$ when acting on $S_N|_{\partial N}$. We note that $(g^*g)^{-\frac{1}{2}}$ is an isometry from $(S_M^*, \langle \cdot, \cdot \rangle)$ to $(S_M^*, \langle \cdot, \cdot \rangle_{sa})$.

More explicitly, at a point $n \in \partial N$, let $\{v_a\}_{a=1}^{2^{\frac{n}{2}}}$ be an orthonormal basis of $S_N$ and let $v_a^*$ denote inner product with respect to $v_a$. Then we put

$$s_C = 2^{-n/2} 1_C (\partial f)^* (g^*g)^{-\frac{1}{2}} \cdot \mathrm{Id} = 2^{-n/2} 1_C \sum_{a=1}^{2^{\frac{n}{2}}} v_a \otimes (\partial f)^* (g^*g)^{-\frac{1}{2}} v_a^*. \tag{3.29}$$

If we divide $\{v_a\}_{a=1}^{2^{\frac{n}{2}}}$ into orthonormal bases for the $\pm 1$-eigenspaces of $\gamma^0 \gamma^n$ then one sees that $T_{sa} s_C = s_C$.

Let $\pi_\pm$ denote orthogonal projections onto the $\pm 1$-eigenspaces of $T_{sa}$ acting on sections of $E|_{\partial N}$. As before, we may have to slightly modify the boundary data $s_C$ to ensure that we can solve the boundary value problem. Put $\bar{s}_C = s_C - P_{\gamma^n \mathrm{Ker}(\mathcal{D})|_{\partial N}} s_C$. Then there is some $\psi \in C^\infty(N; E)$ so that $D^N \psi = 0$ and $\pi_+(\psi|_{\partial N}) = \bar{s}_C$. The section $\psi$ is unique up to addition by elements of $\mathrm{Ker}(\mathcal{D})$. We define $\mathcal{E}_C^{sa}$ as in Definition 2, using $\langle \cdot, \cdot \rangle_{E,sa}$.

*Proposition 15.* If $N$ satisfies the dominant energy condition then $\mathcal{E}_C \geq 0$ and $\mathcal{E}_C^{sa} \geq 0$.
Proof. This follows from Proposition 13. $\square$

*Proposition 16.* $\mathcal{E}_C$ and $\mathcal{E}_C^{sa}$ only depend on $f$ through its boundary value $\partial f$.

*Proof.* The proof is the same as for Proposition 7. $\square$

The quasilocal energy $\mathcal{E}_C$ depends on the choice of constant timelike unit vector field $\mathcal{T}$ in $\mathbb{R}^{n,1}$. By minimizing over such choices, we obtain the quasilocal mass.

*Definition 4.* The quasilocal mass is $\mathcal{M}_C = \inf_{\mathcal{T}} \mathcal{E}_C$.

## D. Background space in a product manifold

In Sec. V, when the background space was a subspace of $\mathbb{R}^{n,1}$, we effectively performed a gauge transformation in order to make $\widehat{\nabla}^W$ locally trivial. This had the effect of making the Dirac-type operator $D^N$ formally self-adjoint on the interior of $N$. Looking at the connection (3.4), one sees that what one really needs is a gauge transformation to make $\widehat{\nabla}^W$ unitary. This can be achieved if the connection on the ambient spinor bundle $S_{\overline{M}}$ is unitary. The easiest situation in which this is guaranteed is when the Lorentzian manifold $\overline{M}$ is an isometric product $\mathbb{R} \times X$ for some $n$-dimensional Riemannian spin manifold $X$. Of course, this includes the case when $\overline{M} = \mathbb{R}^{n,1}$.

Let $M$ be a spacelike hypersurface with boundary in $\overline{M} = \mathbb{R} \times X$. As in Sec. V, we can find local maps $g : U \to \mathrm{Spin}(n)$ on $M$ so that

$$g\left(\frac{1}{8}\widehat{\omega}_{\widehat{\alpha}\widehat{\beta}\widehat{\sigma}}[\widehat{\gamma}^{\widehat{\alpha}}, \widehat{\gamma}^{\widehat{\beta}}] \otimes \widehat{\tau}^{\widehat{\sigma}} - \frac{1}{2}\widehat{\omega}_{\widehat{0}\widehat{\alpha}\widehat{\sigma}}\widehat{\gamma}^{\widehat{0}}\widehat{\gamma}^{\widehat{\alpha}} \otimes \widehat{\tau}^{\widehat{\sigma}}\right)g^{-1} + g\,dg^{-1} \tag{3.30}$$

is a one-form with values in skew-Hermitian matrices (as opposed to its vanishing in Sec. V). We can assume that $g(m_0)$ sends $\widehat{e}_{\widehat{0}}(m_0)$ to the timelike unit vector $\partial_t$ in $T_{m_0}\overline{M}$. On an overlap $U_1 \cap U_2$, the maps $g_1$ and $g_2$ are related by $g_1 = u_{12}g_2\widetilde{\phi}_{12}^{-1}$ for maps $u_{12}, \widetilde{\phi}_{12} : U_1 \cap U_2 \to \mathrm{Spin}(n)$, where $u_{12}$ is covariantly constant and $\widetilde{\phi}_{12}$ comes from the change of local orthonormal frame.

On the domain $U \subset M$, we put $A = g^* g$ and define an inner product $\langle \cdot, \cdot \rangle_{S_M^*, sa}$ by $\langle s_1, s_2 \rangle_{S_M^*, sa} = \langle s_1, As_2 \rangle_{S_M^*}$.

**Proposition 17.** *Suppose that X has nonnegative curvature operator. If $\psi \in C^\infty(N; E)$ satisfies $D^N \psi = 0$ and*

$$2 \left( T_{00} - \sqrt{-\sum_{\alpha=1}^{n} T_{0\alpha} T^{0\alpha}} \right) \geq |\Lambda^2 df|(f^* R_X) \tag{3.31}$$

*then* $-\int_{\partial N} \langle \psi, \nabla_{e_n}^E \psi \rangle \geq 0$.

*Proof.* After making local gauge transformations, the proof is the same as that of Proposition 13. □

We can define $\mathcal{E}_C$ and $\mathcal{E}_C^{sa}$ as in Sec. V. The analogs of Propositions 15 and 16 hold.

## AUTHOR DECLARATIONS

### Conflict of Interest

The author has no conflicts to disclose.

### Author Contributions

**John Lott**: Conceptualization (lead); Formal analysis (lead); Investigation (lead); Methodology (lead); Writing – original draft (lead); Writing – review & editing (lead).

## DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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