SOME OBSTRUCTIONS TO POSITIVE SCALAR CURVATURE ON A NONCOMPACT MANIFOLD

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Abstract. We give obstructions for a noncompact manifold to admit a complete Riemannian metric with (nonuniformly) positive scalar curvature. We treat both the finite volume and infinite volume cases.

1. Introduction

There are many results on whether a given compact manifold admits a Riemannian metric with positive scalar curvature (psc). In this paper we focus on the noncompact case.

An open conjecture for compact manifolds says that an aspherical compact smooth manifold cannot admit a psc metric. There are two main approaches to this conjecture. The first one uses minimal hypersurfaces, following the work of Schoen-Yau [35], and μ-bubbles as introduced by Gromov [17, Section $5\frac{5}{6}$]. Recent advances are by Chodosh-Li [12] and Gromov [18]. The other approach, which we follow, uses Dirac operators.

The above conjecture has an extension to compact manifolds that may not be aspherical. If M is a compact connected oriented n-dimensional smooth manifold, choose a basepoint m_0 and consider the fundamental group $\Gamma = \pi_1(M, m_0)$. There is a pointed connected CW-complex $B\Gamma$ with the property that $\pi_1(B\Gamma) = \Gamma$ and the universal cover of $B\Gamma$ is contractible. There is a classifying map $\nu : M \to B\Gamma$, unique up to homotopy, that induces an isomorphism on π_1. If $[M] \in \mathrm{H}_n(M; \mathbb{Q})$ is the fundamental class in rational homology then the extended conjecture says that nonvanishing of the pushforward $\nu_*[M] \in \mathrm{H}_n(B\Gamma; \mathbb{Q})$ is an obstruction for M to admit a psc metric. If M is aspherical then one recovers the previous conjecture. There are many results on this extended conjecture, using Dirac operators [32].

This paper is concerned with obstructions to complete psc metrics on noncompact manifolds. There is also a long history to this problem, going back to the Gromov-Lawson paper [20]. There is a technical difference between looking at metrics with uniformly positive scalar curvature, i.e. bounded below by a positive constant, and metrics with nonuniformly positive scalar curvature. In the first case the Dirac operator is Fredholm, while in the second case it need not be Fredholm. We are interested in psc metrics that may not have uniformly positive scalar curvature; the study of such metrics goes back to [20, Section 6]. Our motivation comes from conjectures relating scalar curvature to simplicial volume, for compact manifolds.

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Here is a test question. Suppose that Y is a compact connected oriented n-dimensional manifold-with-boundary, with connected boundary ∂Y. Choosing a basepoint $y_0 \in \partial Y$, put $\Gamma = \pi_1(Y, y_0)$ and $\Gamma' = \pi_1(\partial Y, y_0)$. There is a classifying map of pairs $\nu : (Y, \partial Y) \to (B\Gamma, B\Gamma')$, unique up to homotopy. Let $[Y, \partial Y] \in \mathrm{H}_n(Y, \partial Y; \mathbb{Q})$ be the fundamental class. Is nonvanishing of the pushforward $\nu_*[Y, \partial Y] \in \mathrm{H}_n(B\Gamma, B\Gamma'; \mathbb{Q})$ an obstruction for the interior $\mathrm{int}(Y) = Y - \partial Y$ of Y to admit a complete psc metric, provided that
(a) The homomorphism $\Gamma' \to \Gamma$ is injective, or
(b) The metric has finite volume?

Here if $\Gamma' \to \Gamma$ is not injective then we define $\mathrm{H}_n(B\Gamma, B\Gamma'; \mathbb{Q})$ using the algebraic mapping cone complex; it could more accurately be written as $\mathrm{H}_n(B\Gamma' \to B\Gamma; \mathbb{Q})$.

One needs some condition like (a) or (b), as can be seen if $Y = D^2$. Then $\pi_1(\partial Y) \to \pi_1(Y)$ is not injective, $\nu_*[Y, \partial Y] \neq 0$ and $\mathrm{int}(Y)$ does admit a complete psc metric, such as a paraboloid, but not one of finite volume.

A special case of the above question is when Y and ∂Y are aspherical, in which case $\nu_*[Y, \partial Y]$ is automatically nonzero.

We give results in the direction of (a) and (b). One main tool is almost flat bundles in the relative setting. Almost flat bundles were introduced by Connes-Gromov-Moscovici [13] and give obstructions for compact spin manifolds to have psc metrics. We review this material in Section 2. Almost flat bundles in the relative setting were introduced by Kubota [26]. There are actually two versions: almost flat relative bundles and almost flat stable relative bundles. They are relevant for (a) and (b), respectively.

Our other main technical tool is Callias-type Dirac operators, as were used for example by Cecchini-Zeidler in [8, 9]. This allows us to give localized obstructions to positive scalar curvature, that apply to incomplete manifolds. Some of the statements involve the mean curvature of a boundary.

For notation, R denotes scalar curvature. Our convention for mean curvature is such that $S^{n-1} \subset D^n$ has mean curvature $H = n - 1$.

The geometric setup for our localized statements is that we have a region in a manifold that is assumed to have positive scalar curvature, then there is an annulus around it with quantitatively positive scalar curvature, followed by a larger annulus of a certain size that can have some negative scalar curvature. What happens outside of the second annulus doesn't matter. The precise assumption is the following.

Assumption 1.1. *Given $r_0, D > 0$, put $r_0' = \frac{1}{256} r_0^2 D^2$ and $D' = D + \frac{32}{r_0 D}$. Let M be a connected Riemannian spin manifold, possibly with boundary and possibly incomplete. Let K be a compact subset of M containing ∂M. Suppose that*

- *The distance neighborhood $N_{D'}(K)$ lies in a compact submanifold-with-boundary $\mathcal{C}$,*
- *$R > 0$ on K,*
- *$R \geq r_0$ on $N_D(K) - K$ and*
- *$R \geq -r_0'$ on $N_{D'}(K) - N_D(K)$.*

The first main result just uses almost flat K-theory classes, denoted by $\mathrm{K}^*_{af}(\cdot)$.

Theorem 1.2. *Suppose that Assumption 1.1 holds, where ∂M has nonnegative mean curvature. Given $\beta \in \mathrm{K}^{-1}_{af}(\mathcal{C})$ if M is even dimensional, or $\beta \in \mathrm{K}^{0}_{af}(\mathcal{C})$ if M is odd dimensional, we have*

$$\int_{\partial M} \widehat{A}(T\partial M) \wedge \mathrm{ch}\left(\beta\big|_{\partial M}\right) = 0. \tag{1.3}$$

Our main application of Theorem 1.2 is to finite volume complete manifolds M of dimension at most seven. Proposition 2.40 says that there is an exhaustion of M by compact submanifolds K_i whose boundaries ∂K_i have nonnegative mean curvature as seen from $M - K_i$. (It would be interesting if the dimension restriction could be removed.) Rather than applying Theorem 1.2 to K_i, we apply it to a suitable compact region of $\overline{M - K_i}$ containing ∂K_i. In this way we obtain end obstructions to the existence of finite volume psc metrics. As a simple example, there is no complete finite volume psc metric on $[0, \infty) \times T^{n-1}$ if $n \leq 7$.

The next main result uses almost flat relative K-theory classes, denoted by $\mathrm{K}^*_{af}(\cdot, \cdot)$. The geometric setup is similar to the previous one, except that now there is no boundary.

Theorem 1.4. *Suppose that Assumption 1.1 holds, where $\partial M = \emptyset$ and K is a compact submanifold-with-boundary in M. Then given $\beta \in \mathrm{K}^*_{af}\left(\mathcal{C}, \mathcal{C} - \mathrm{int}(K)\right)$, we have*

$$\int_{\mathcal{C}} \widehat{A}(TM) \wedge \mathrm{ch}(\beta) = 0. \tag{1.5}$$

Theorem 1.4 is relevant to part (a) of the test question above; see Corollary 3.41.

The third main result combines Theorems 1.2 and 1.4. It uses almost flat stable relative K-theory classes, denoted by $\mathrm{K}^*_{af,st}(\cdot, \cdot)$ The generators of $\mathrm{K}^0_{af,st}(\cdot, \cdot)$ differ from generators of $\mathrm{K}^0_{af}(\cdot, \cdot)$ essentially by having additional K^{-1}-generators for the second factor. This meshes well with the compact exhaustions of finite volume manifolds. In the geometric assumptions, it is now assumed that the boundary of the inner compact region has nonnegative mean curvature as seen from the complement.

Theorem 1.6. *Suppose that Assumption 1.1 holds, where $\partial M = \emptyset$, K is a compact codimension-zero submanifold-with-boundary in M, and ∂K has nonnegative mean curvature as seen from $M - K$. Then given $\beta \in \mathrm{K}^*_{af,st}\left(\mathcal{C}, \mathcal{C} - \mathrm{int}(K)\right)$, we have*

$$\int_{\mathcal{C}} \widehat{A}(TM) \wedge \mathrm{ch}(\beta) = 0. \tag{1.7}$$

When combined with the result about compact exhaustions of complete finite volume Riemannian manifolds, Theorem 1.6 is relevant to part (b) of the test question above; see Corollary 4.15.

For technical reasons, our results use various types of almost flat bundles. Another approach toward index theoretic obstructions to positive scalar curvature uses group C^*-algebras. It would be interesting if similar results could be proved using relative C^*-algebras and Mishchenko bundles.

Our interest in psc metrics comes from a conjecture about the simplicial volume of manifolds having almost nonnegative scalar curvature, with respect to a normalized volume.

We wanted to see if the conjecture can be verified under additional geometric bounds. We were only partly successful in this; see Appendix A.

The structure of the paper is the following. In Section 2 we prove Theorem 1.2 and give an application to finite volume Riemannian manifolds. In Section 3 we prove Theorem 1.4. We also give obstructions for a compact manifold-with-boundary to have a psc metric with nonnegative mean curvature on the boundary. In Section 4 we prove Theorem 1.6. The appendix has a discussion of simplicial volume. More detailed descriptions are at the beginnings of the sections.

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2. Almost flat bundles and end obstructions

In this section we use almost flat bundles to construct obstructions for noncompact manifolds to admit complete finite volume psc metrics. In Subsection 2.1 we review almost flat bundles and their use for compact manifolds. Subsection 2.2 gives an obstruction for a manifold-with-boundary to have a metric with positive scalar curvature in a neighborhood of the boundary, with nonnegative mean curvature on the boundary. In Subsection 2.3 we give obstructions for a manifold to have a complete finite volume psc metric.

2.1. **Almost flat bundles.**

If X is a compact topological space then generators of $\mathrm{K}^0(X)$ are $\mathbb{Z}_2$-graded vector bundles on X, and there is a Chern character $\mathrm{ch} : \mathrm{K}^0(X) \to \mathrm{H}^{even}(X;\mathbb{Q})$. Generators of $\mathrm{K}^{-1}(X)$ are given by pairs (V, σ) where V is a vector bundle on X and σ is an automorphism of V. There is a Chern character $\mathrm{ch} : \mathrm{K}^{-1}(X) \to \mathrm{H}^{odd}(X;\mathbb{Q})$.

If X is noncompact then we take $\mathrm{K}(X)$ to be the representable K-group, i.e. $\mathrm{K}^0(X) = [X, \mathbb{Z} \times BU]$. In what follows, all manifolds will be taken to be smooth.

Definition 2.1. [13] *Let $\mathcal{N}$ be a compact manifold-with-boundary. Put a Riemannian metric on $\mathcal{N}$. Given $\beta \in \mathrm{K}^0(\mathcal{N})$ we say that β is almost flat if for every $\epsilon > 0$, we can find*

- *A $\mathbb{Z}_2$-graded Hermitian vector bundle V on $\mathcal{N}$ representing β, and*
- *A Hermitian connection $\nabla^{\pm}$ on V whose curvature satisfies $\|F^{\pm}\| \leq \epsilon$.*

Whether β is almost flat is independent of the choice of Riemannian metric on $\mathcal{N}$. Let $\mathrm{K}^0_{af}(\mathcal{N})$ denote the almost flat elements of $\mathrm{K}^0(\mathcal{N})$. The relevance of almost flat bundles for questions of positive scalar curvature comes from the following result.

Theorem 2.2. [13] *If M is a closed even dimensional spin manifold with a positive scalar curvature metric, and $\beta \in \mathrm{K}^0_{af}(M)$, then $\int_M \widehat{A}(TM) \wedge \mathrm{ch}(\beta) = 0$.*

Recall that a discrete group Γ has a classifying space $B\Gamma$ which can be taken to be a connected CW-complex, defined up to homotopy, with the property that $\pi_1(B\Gamma) \cong \Gamma$ and the universal cover of $B\Gamma$ is contractible. If $\mathcal{N}$ is connected, let $\nu_{\mathcal{N}} : \mathcal{N} \to B\pi_1(\mathcal{N}, n_0)$ be the classifying map for the universal cover of $\mathcal{N}$. It is known that elements of $\mathrm{K}^0_{af}(\mathcal{N})$ pull back from $B\pi_1(\mathcal{N}, n_0)$, in an appropriate sense [22, 31]. This motivates the following definition.

Definition 2.3. *Let Γ be a discrete group. Given $\eta \in \mathrm{K}^0(B\Gamma)$, we say that η is almost flat if for every compact manifold-with-boundary $\mathcal{N}$ and every continuous map $\nu : \mathcal{N} \to B\Gamma$, the pullback $\nu^*\eta$ is almost flat.*

Let $\mathrm{K}^0_{af}(B\Gamma)$ denote the almost flat classes. It is a subgroup of $\mathrm{K}^0(B\Gamma)$. One can give a more intrinsic description of $\mathrm{K}^0_{af}(B\Gamma)$ in terms of Γ [22]. It seems conceivable that for every discrete group Γ, the map $\mathrm{K}^0_{af}(B\Gamma) \otimes \mathbb{Q} \to \mathrm{K}^0(B\Gamma) \otimes \mathbb{Q}$ is an isomorphism, at least if $B\Gamma$ can be represented by a finite CW-complex. (Our definitions avoid some subtleties in the case of infinite complexes.) Some examples of groups Γ where this is known are listed in [13].

Note that if $B\Gamma$ can be represented by a smooth compact manifold then $\eta \in \mathrm{K}^0(B\Gamma)$ is almost flat in the sense of Definition 2.3 if and only if it is almost flat in the sense of Definition 2.1.

We now give the odd degree version of the previous definitions.

Definition 2.4. *Let $\mathcal{N}$ be a compact manifold-with-boundary. Put a Riemannian metric on $\mathcal{N}$. Given $\beta \in \mathrm{K}^{-1}(\mathcal{N})$, we say that β is almost flat if for every $\epsilon > 0$, we can find*

- *A Hermitian vector bundle V on $\mathcal{N}$ and a unitary automorphism σ of V so that (V, σ) represents β, and*
- *A Hermitian connection ∇ on V whose curvature satisfies $\|F\| \leq \epsilon$, and also $\|\nabla\sigma\| \leq \epsilon$.*

Whether β is almost flat is independent of the choice of Riemannian metric on $\mathcal{N}$. Let $\mathrm{K}^{-1}_{af}(\mathcal{N})$ denote the almost flat elements of $\mathrm{K}^{-1}(\mathcal{N})$.

Definition 2.5. *Let Γ be a discrete group. Given $\eta \in \mathrm{K}^{-1}(B\Gamma)$, we say that η is almost flat if for every compact manifold-with-boundary $\mathcal{N}$ and every continuous map $\nu : \mathcal{N} \to B\Gamma$, the pullback $\nu^*\eta$ is almost flat.*

Let $\mathrm{K}^{-1}_{af}(B\Gamma)$ denote the almost flat classes. It is a subgroup of $\mathrm{K}^{-1}(B\Gamma)$.

The generator of $\mathrm{K}^{-1}(S^1) \cong \mathbb{Z}$ is almost flat [36, Exemple 3.2]. There is a product $\mathrm{K}^*(B\Gamma) \times \mathrm{K}^{-1}(S^1) \to \mathrm{K}^{*-1}(B\Gamma \times S^1)$ that restricts to a product $\mathrm{K}^*_{af}(B\Gamma) \times \mathrm{K}^{-1}_{af}(S^1) \to \mathrm{K}^{*-1}_{af}(B\Gamma \times S^1)$.

Corollary 2.6. *Given*

- *A closed spin manifold M with a positive scalar curvature metric,*
- *A discrete group Γ,*
- *A continuous map $\nu : M \to B\Gamma$, and*
- *An element $\eta \in \mathrm{K}^*_{af}(B\Gamma)$,*

we have $\int_M \widehat{A}(TM) \wedge \mathrm{ch}(\nu^*\eta) = 0$.

Corollary 2.7. *Given a closed connected spin manifold M with a positive scalar curvature metric, suppose that $\mathrm{K}^*_{af}(B\pi_1(M, m_0)) \otimes \mathbb{Q} = \mathrm{K}^*(B\pi_1(M, m_0)) \otimes \mathbb{Q}$. Let $[M] \in \mathrm{H}_{\dim(M)}(M; \mathbb{Q})$ be the fundamental class. Let $\nu : M \to B\pi_1(M, m_0)$ be the classifying map. Then $\nu_*[M]$ vanishes in $\mathrm{H}_{\dim(M)}(B\pi_1(M, m_0)); \mathbb{Q})$.*

Proof. Given $\eta \in \mathrm{K}^*(B\pi_1(M, m_0))$, we have $\int_M \widehat{A}(TM) \wedge \mathrm{ch}(\nu^*\eta) = \langle \nu_*(\star\widehat{A}(TM)), \mathrm{ch}(\eta)\rangle$, where $\star\widehat{A}(TM) \in \mathrm{H}_*(M;\mathbb{Q})$ is the Poincaré dual of $\widehat{A}(TM) \in \mathrm{H}^*(M;\mathbb{Q})$. Under the assumptions of the corollary, $\nu_*(\star\widehat{A}(TM))$ vanishes in $\mathrm{H}_*(B\pi_1(M, m_0);\mathbb{Q})$. As $\star(1) = [M]$, the corollary follows. □

2.2. **Boundary obstruction.** The next result gives an obstruction for a manifold-with-boundary to have a metric with positive scalar curvature in a neighborhood of the boundary, with nonnegative mean curvature on the boundary. We will use the following running assumption in this subsection.

Assumption 2.8. *Given $r_0, D > 0$, put $r_0' = \frac{1}{256} r_0^2 D^2$ and $D' = D + \frac{32}{r_0 D}$. Let M be a connected Riemannian spin manifold-with-boundary, possibly incomplete. Suppose that ∂M is compact and has nonnegative mean curvature. Let K be a compact subset of M containing ∂M.*

Suppose that

- *The distance neighborhood $N_{D'}(K)$ lies in a compact submanifold-with-boundary $\mathcal{C}$,*
- *$R > 0$ on K,*
- *$R \geq r_0$ on $N_D(K) - K$ and*
- *$R \geq -r_0'$ on $N_{D'}(K) - N_D(K)$.*

Theorem 2.9. *Suppose that Assumption 2.8 holds. Given $\beta \in \mathrm{K}^{-1}_{af}(\mathcal{C})$ if M is even dimensional, or $\beta \in \mathrm{K}^0_{af}(\mathcal{C})$ if M is odd dimensional, we have*

$$\int_{\partial M} \widehat{A}(T\partial M) \wedge \mathrm{ch}\left(\beta\big|_{\partial M}\right) = 0. \tag{2.10}$$

Proof. Suppose first that M is even dimensional. Let $\epsilon > 0$ be a small parameter, which we will adjust. Let $d_K \in C(\mathcal{C})$ be the distance function from K. We would like to use level sets of d_K but they need not be smooth. To get around this, let $\widetilde{d}_K$ be a slight smoothing of d_K on $N_{D'}(K)$ so that $|\widetilde{d}_K - d_K| < \epsilon$ and $|\nabla \widetilde{d}_K| < 1 + \epsilon$. Put $\lambda_1 = D - \epsilon$. Choose a regular value λ_2 of $\widetilde{d}_K$ that is slightly greater than $D - \epsilon + \frac{16}{r_0(D-2\epsilon)}$. Put $\lambda_0 = \lambda_1 - \frac{16}{r_0(\lambda_2 - \lambda_1)}$. Then λ_0 is slightly greater than ϵ. By construction,

$$\begin{aligned} \widetilde{d}_K^{-1}(-\infty, \lambda_0) &\subset N_{3\epsilon}(K), \\ \widetilde{d}_K^{-1}(\lambda_0, \lambda_1) &\subset N_D(K) - K, \\ \widetilde{d}_K^{-1}(\lambda_1, \lambda_2) &\subset N_{D'}(K). \end{aligned} \tag{2.11}$$

In particular,

$$\begin{cases} R(x) > 0 & \text{if } \widetilde{d}_K(x) \leq \lambda_0, \\ R(x) \geq r_0 & \text{if } \lambda_0 \leq \widetilde{d}_K(x) \leq \lambda_1, \\ R(x) \geq -r_0' & \text{if } \lambda_1 \leq \widetilde{d}_K(x) < \lambda_2. \end{cases} \tag{2.12}$$

Define $\sigma \in C(-\infty, \lambda_2)$ by

$$\sigma(t) = \begin{cases} 0 & \text{if } t \leq \lambda_0, \\ \frac{r_0}{8}(t-\lambda_0) & \text{if } \lambda_0 \leq t \leq \lambda_1, \\ \frac{2}{\lambda_2 - t} & \text{if } \lambda_1 \leq t < \lambda_2. \end{cases} \tag{2.13}$$

Let $\widetilde{\sigma}$ be a slight smoothing of σ. Put $F = \sigma \circ \widetilde{d}_K$ and $f = \widetilde{\sigma} \circ \widetilde{d}_K$.

The preimage $\widetilde{d}_K^{-1}(-\infty, \lambda_2]$ is a smooth compact manifold-with-boundary. Its boundary is the disjoint union of ∂M and $\widetilde{d}_K^{-1}(\lambda_2)$. For small ϵ, let $\mathcal{N}^\epsilon$ be the codimension-zero submanifold of $\widetilde{d}_K^{-1}(-\infty, \lambda_2]$ whose boundary consists of ∂M along with interior points of distance ϵ from $\widetilde{d}_K^{-1}(\lambda_2)$. We write $\partial \mathcal{N}^\epsilon = \partial M \cup \partial_+ \mathcal{N}^\epsilon$.

Let $\beta^\epsilon \in \mathrm{K}^{-1}_{af}(\mathcal{N}^\epsilon)$ be the restriction of β to $\mathcal{N}^\epsilon$. As in Definition 2.4, let (V, σ) be a Hermitian vector bundle on $\mathcal{N}^\epsilon$ with unitary automorphism that represents β^ϵ. Let E be the $\mathbb{Z}_2$-graded vector bundle $V \oplus V$. The operator $\begin{pmatrix} 0 & \sigma^{-1} \\ \sigma & 0 \end{pmatrix}$ is an odd self-adjoint endomorphism of E. Put

$$A = \begin{pmatrix} \nabla^V & f\sigma^{-1} \\ f\sigma & \nabla^V \end{pmatrix}, \tag{2.14}$$

a superconnection on E. Let D^E be the quantization of A, i.e.

$$D^E = \begin{pmatrix} D^V & \epsilon_S f \sigma^{-1} \\ \epsilon_S f \sigma & D^V \end{pmatrix}, \tag{2.15}$$

where ϵ_S is the $\mathbb{Z}_2$-grading operator on the spinor bundle S and D^V is the Dirac operator on $C^\infty(M; S \otimes V)$. Then

$$\begin{aligned} (D^E)^2 = & \begin{pmatrix} (D^V)^2 + f^2 & 0 \\ 0 & (D^V)^2 + f^2 \end{pmatrix} + \\ & \sqrt{-1}\epsilon_S c(df) \begin{pmatrix} 0 & \sigma^{-1} \\ \sigma & 0 \end{pmatrix} + \sqrt{-1}\epsilon_S f \begin{pmatrix} 0 & -\sigma^{-1} c(\nabla^V \sigma) \sigma^{-1} \\ c(\nabla^V \sigma) & 0 \end{pmatrix}, \end{aligned} \tag{2.16}$$

where $c(w)$ denotes Clifford multiplication by a 1-form w (satisfying $c(w)^2 = |w|^2$).

For notation, if $\{e_\alpha\}_{\alpha=1}^n$ is a local orthonormal frame of TM and $\{\tau^\alpha\}_{\alpha=1}^n$ is the dual coframe then we put $\gamma^\alpha = c(\tau^\alpha)$. Locally, $D^V = -\sqrt{-1} \sum_{\alpha=1}^n \gamma^\alpha \nabla_{e_\alpha}$.

We take the inner product $\langle \psi_1, \psi_2 \rangle$ on $S \otimes E$ to be $\mathbb{C}$-linear in ψ_2 and $\mathbb{C}$-antilinear in ψ_1. On $\partial \mathcal{N}^\epsilon$, we let e_n be the inward pointing normal vector. For $\psi_1, \psi_2 \in C^\infty(\mathcal{N}^\epsilon; S \otimes E)$, we have

$$\begin{aligned} & \int_{\mathcal{N}^\epsilon} \langle D^E \psi_1, \psi_2 \rangle \, \mathrm{dvol}_{\mathcal{N}^\epsilon} - \int_{\mathcal{N}^\epsilon} \langle \psi_1, D^E \psi_2 \rangle \, \mathrm{dvol}_{\mathcal{N}^\epsilon} = \\ & - \sqrt{-1} \int_{\partial \mathcal{N}^\epsilon} \langle \psi_1, \gamma^n \psi_2 \rangle \, \mathrm{dvol}_{\partial \mathcal{N}^\epsilon} . \end{aligned} \tag{2.17}$$

Define a self-adjoint operator Π on $C^\infty\left(\partial\mathcal{N}^\epsilon, (S\otimes E)\big|_{\partial\mathcal{N}^\epsilon}\right)$ by

$$\Pi = \begin{cases} \begin{pmatrix} 0 & \sqrt{-1}\epsilon_S\gamma^n \\ \sqrt{-1}\epsilon_S\gamma^n & 0 \end{pmatrix} & \text{on } \partial M, \\ \begin{pmatrix} 0 & \sqrt{-1}\epsilon_S\gamma^n\sigma^{-1} \\ \sqrt{-1}\epsilon_S\gamma^n\sigma & 0 \end{pmatrix} & \text{on } \partial_+\mathcal{N}^\epsilon \end{cases}. \tag{2.18}$$

Then $\Pi^2 = \mathrm{Id}$, $\Pi\gamma^n + \gamma^n\Pi = 0$ and Π commutes with the $\mathbb{Z}_2$ grading on $(S\otimes E)\big|_{\partial\mathcal{N}^\epsilon}$. It follows from (2.17) that with the boundary condition $\Pi\psi = \psi$, the operator D^E on $C^\infty(\mathcal{N}^\epsilon; S\otimes E)$ is formally self-adjoint. In fact, the boundary condition is an elliptic boundary condition [2, Section 7.5]. We let $\mathcal{D}$ denote the ensuing self-adjoint operator, densely defined on $L^2(\mathcal{N}^\epsilon; S\otimes E)$.

Lemma 2.19. *There is an $\epsilon_0 > 0$ so that if $\epsilon < \epsilon_0$, and in addition $\|F^V\| \le \epsilon$ and $\|\nabla^V\sigma\| \le \epsilon$, then the kernel of $\mathcal{D}$ vanishes.*

Proof. One can check that

$$|\nabla F|(x) = \begin{cases} 0 & \text{if } \widetilde{d}_K(x) < \lambda_0, \\ \frac{r_0}{8}|\nabla\widetilde{d}_K|(x) & \text{if } \lambda_0 < \widetilde{d}_K(x) < \lambda_1, \\ \frac{2}{(\lambda_2-\widetilde{d}_K(x))^2}|\nabla\widetilde{d}_K|(x) & \text{if } \lambda_1 < \widetilde{d}_K(x) < \lambda_2. \end{cases} \tag{2.20}$$

Then

$$|\nabla F|(x) \le \begin{cases} 0 & \text{if } \widetilde{d}_K(x) < \lambda_0, \\ \frac{r_0}{8}(1+\epsilon) & \text{if } \lambda_0 < \widetilde{d}_K(x) < \lambda_1, \\ \frac{2}{(\lambda_2-\widetilde{d}_K(x))^2}(1+\epsilon) & \text{if } \lambda_1 < \widetilde{d}_K(x) < \lambda_2. \end{cases} \tag{2.21}$$

Writing $F^2 - \epsilon F = \left(F - \frac{\epsilon}{2}\right)^2 - \frac{\epsilon^2}{4}$, we have

$$(F^2 - \epsilon F)(x) \ge \begin{cases} 0 & \text{if } \widetilde{d}_K(x) < \lambda_0, \\ -\frac{\epsilon^2}{4} & \text{if } \lambda_0 < \widetilde{d}_K(x) < \lambda_1, \\ \left(\frac{2}{\lambda_2-\widetilde{d}_K(x)} - \frac{\epsilon}{2}\right)^2 - \frac{\epsilon^2}{4} & \text{if } \lambda_1 < \widetilde{d}_K(x) < \lambda_2. \end{cases} \tag{2.22}$$

For small ϵ, we can bound $\left(\frac{2}{\lambda_2-\widetilde{d}_K(x)} - \frac{\epsilon}{2}\right)^2 - \frac{2}{(\lambda_2-\widetilde{d}_K(x))^2}(1+\epsilon)$ below by $\frac{1}{(\lambda_2-\widetilde{d}_K(x))^2}$. Using the fact that $\lambda_2 - \lambda_1 = \frac{16}{r_0(D-2\epsilon)}$, it follows that for small ϵ, we have

$$(F^2 - |\nabla F| - \epsilon F)(x) \ge \begin{cases} 0 & \text{if } \widetilde{d}_K(x) < \lambda_0, \\ -\frac{\epsilon^2}{4} - \frac{r_0}{8}(1+\epsilon) & \text{if } \lambda_0 < \widetilde{d}_K(x) < \lambda_1, \\ \frac{r_0^2 D^2}{512} & \text{if } \lambda_1 < \widetilde{d}_K(x) < \lambda_2. \end{cases} \tag{2.23}$$

Combining with (2.12), we see that for small ϵ, the function $\frac{R}{4} + F^2 - |\nabla F| - \epsilon F$ is bounded below on its domain in $\widetilde{d}_K^{-1}(-\infty, \lambda_2]$ by a positive constant that is independent of ϵ. Taking $\widetilde{\sigma}$ to be an appropriate smoothing of σ, the same will be true of $\frac{R}{4} + f^2 - |\nabla f| - \epsilon f$.

The Lichnerowicz formula gives

$$(D^V)^2 = \nabla^*\nabla + \frac{R}{4} - \frac{1}{4}[\gamma^\mu, \gamma^\nu] F^V_{\mu\nu}. \tag{2.24}$$

If $\psi \in C^\infty(\mathcal{N}^\epsilon; S \otimes E)$ then

$$\int_{\mathcal{N}^\epsilon} \langle \psi, \nabla^*\nabla\psi \rangle \ \mathrm{dvol}_{\mathcal{N}^\epsilon} = \int_{\mathcal{N}^\epsilon} |\nabla\psi|^2 \ \mathrm{dvol}_{\mathcal{N}^\epsilon} + \int_{\partial\mathcal{N}^\epsilon} \langle \psi, \nabla_{e_n}\psi \rangle \ \mathrm{dvol}_{\partial\mathcal{N}^\epsilon}. \tag{2.25}$$

Suppose that ψ is a nonzero solution of $D^E\psi = 0$. Then $\int_{\mathcal{N}^\epsilon} \langle \psi, (D^E)^2\psi \rangle \ \mathrm{dvol}_{\mathcal{N}^\epsilon} = 0$, which from (2.16) becomes

$$\begin{aligned} 0 = &\int_{\mathcal{N}^\epsilon} \left(|\nabla\psi|^2 + \left(\frac{R}{4} + f^2 \right) |\psi|^2 - \frac{1}{4} \langle \psi, [\gamma^\alpha, \gamma^\beta] F^V_{\alpha\beta} \psi \rangle \right) \ \mathrm{dvol}_{\mathcal{N}^\epsilon} + \\ &\int_{\mathcal{N}^\epsilon} \left\langle \psi, \sqrt{-1}\epsilon_S c(df) \begin{pmatrix} 0 & \sigma^{-1} \\ \sigma & 0 \end{pmatrix} \psi \right\rangle \mathrm{dvol}_{\mathcal{N}^\epsilon} + \\ &\int_{\mathcal{N}^\epsilon} \left\langle \psi, \sqrt{-1}\epsilon_S f \begin{pmatrix} 0 & -\sigma^{-1} c(\nabla^V \sigma) \sigma^{-1} \\ c(\nabla^V \sigma) & 0 \end{pmatrix} \right\rangle \mathrm{dvol}_{\mathcal{N}^\epsilon} + \\ &\int_{\partial\mathcal{N}^\epsilon} \langle \psi, \nabla_{e_n}\psi \rangle \ \mathrm{dvol}_{\partial\mathcal{N}^\epsilon}. \end{aligned} \tag{2.26}$$

From $D^E\psi = 0$, one can solve for $\nabla_{e_n}\psi$ and obtain

$$\begin{aligned} \int_{\partial\mathcal{N}^\epsilon} \langle \psi, \nabla_{e_n}\psi \rangle \ \mathrm{dvol}_{\partial\mathcal{N}^\epsilon} = &\int_{\partial\mathcal{N}^\epsilon} \langle \psi, D^E_{\partial\mathcal{N}^\epsilon}\psi \rangle \ \mathrm{dvol}_{\partial\mathcal{N}^\epsilon} + \int_{\partial\mathcal{N}^\epsilon} H|\psi|^2 \ \mathrm{dvol}_{\partial\mathcal{N}^\epsilon} + \\ &\int_{\partial\mathcal{N}^\epsilon} \left\langle \psi, \begin{pmatrix} 0 & \sqrt{-1} f \epsilon_S \gamma^n \sigma^{-1} \\ \sqrt{-1} f \epsilon_S \gamma^n \sigma & 0 \end{pmatrix} \psi \right\rangle \ \mathrm{dvol}_{\partial\mathcal{N}^\epsilon}, \end{aligned} \tag{2.27}$$

where

$$D^E_{\partial\mathcal{N}^\epsilon} = - \sum_{j=1}^{n-1} \gamma^n \gamma^j \nabla_{e_j} \otimes I_2 \tag{2.28}$$

is the intrinsic Dirac operator on $\partial\mathcal{N}^\epsilon$ coupled to $E\Big|_{\partial\mathcal{N}^\epsilon}$.

We now impose the boundary condition $\Pi\psi = \psi$. Then (2.27) becomes

$$\int_{\partial\mathcal{N}^\epsilon} \langle \psi, \nabla_{e_n}\psi \rangle \ \mathrm{dvol}_{\partial\mathcal{N}^\epsilon} = \int_{\partial\mathcal{N}^\epsilon} \langle \psi, D^E_{\partial\mathcal{N}^\epsilon}\psi \rangle \ \mathrm{dvol}_{\partial\mathcal{N}^\epsilon} + \int_{\partial\mathcal{N}^\epsilon} (H + f)|\psi|^2 \ \mathrm{dvol}_{\partial\mathcal{N}^\epsilon}. \tag{2.29}$$

On ∂M, we have $D^E_{\partial\mathcal{N}^\epsilon}\Pi + \Pi D^E_{\partial\mathcal{N}^\epsilon} = 0$, which implies that $\int_{\partial M} \langle \psi, D^E_{\partial\mathcal{N}^\epsilon}\psi \rangle \ \mathrm{dvol}_{\partial M} = 0$. On $\partial_+\mathcal{N}^\epsilon$ we have

$$D^E_{\partial\mathcal{N}^\epsilon}\Pi + \Pi D^E_{\partial\mathcal{N}^\epsilon} = \begin{pmatrix} 0 & -\sqrt{-1}\epsilon_S \gamma^j \sigma^{-1} (\nabla^V_{e_j}\sigma) \sigma^{-1} \\ \sqrt{-1}\epsilon_S \gamma^j \nabla^V_{e_j}\sigma & 0 \end{pmatrix}, \tag{2.30}$$

which gives

$$\int_{\partial_+\mathcal{N}^\epsilon} \langle \psi, D^E_{\partial\mathcal{N}^\epsilon}\psi\rangle \ \mathrm{dvol}_{\partial_+\mathcal{N}^\epsilon} = \frac{1}{2}\int_{\partial_+\mathcal{N}^\epsilon} \langle \psi, \left(D^E_{\partial\mathcal{N}^\epsilon}\Pi + \Pi D^E_{\partial\mathcal{N}^\epsilon}\right)\psi\rangle \ \mathrm{dvol}_{\partial_+\mathcal{N}^\epsilon} = \tag{2.31}$$
$$\frac{1}{2}\int_{\partial_+\mathcal{N}^\epsilon} \left\langle \psi, \begin{pmatrix} 0 & -\sqrt{-1}\epsilon_S\gamma^j\sigma^{-1}(\nabla^V_{e_j}\sigma)\sigma^{-1} \\ \sqrt{-1}\epsilon_S\gamma^j\nabla^V_{e_j}\sigma & 0\end{pmatrix}\psi\right\rangle \ \mathrm{dvol}_{\partial_+\mathcal{N}^\epsilon}.$$

Hence

$$\int_{\partial_+\mathcal{N}^\epsilon} \langle \psi, \nabla_{e_n}\psi\rangle \ \mathrm{dvol}_{\partial_+\mathcal{N}^\epsilon} \ge \tag{2.32}$$
$$\int_{\partial M} H|\psi|^2 \, \mathrm{dvol}_{\partial M} + \int_{\partial_+\mathcal{N}^\epsilon} (H + f - \mathrm{const.}\, \epsilon)|\psi|^2 \ \mathrm{dvol}_{\partial_+\mathcal{N}^\epsilon},$$

where const. = const.(n).

Referring to the terms in (2.26), we have a bound

$$\int_{\mathcal{N}^\epsilon}\left(\frac{R}{4} + f^2\right)|\psi|^2 \ \mathrm{dvol}_{\mathcal{N}^\epsilon} + \int_{\mathcal{N}^\epsilon}\left\langle \psi, \sqrt{-1}\epsilon_S c(df)\begin{pmatrix} 0 & \sigma^{-1} \\ \sigma & 0\end{pmatrix}\psi\right\rangle \mathrm{dvol}_{\mathcal{N}^\epsilon} + \tag{2.33}$$
$$\int_{\mathcal{N}^\epsilon}\left\langle \psi, \sqrt{-1}\epsilon_S f\begin{pmatrix} 0 & -\sigma^{-1}c(\nabla^V\sigma)\sigma^{-1} \\ c(\nabla^V\sigma) & 0\end{pmatrix}\right\rangle \mathrm{dvol}_{\mathcal{N}^\epsilon} \ge$$
$$\int_{\mathcal{N}^\epsilon}\left(\frac{R}{4} + f^2 - |\nabla f| - \epsilon f\right)|\psi|^2 \ \mathrm{dvol}_{\mathcal{N}^\epsilon}.$$

Since $\widetilde{d}_K^{-1}(\lambda_2)$ is a smooth compact hypersurface in M, for small ϵ there is a uniform bound on the mean curvature H of $\partial_+\mathcal{N}^\epsilon$, independent of ϵ. As f blows up near $\widetilde{d}_K^{-1}(\lambda_2)$, equation (2.32) implies that for small ϵ the expression $\int_{\partial\mathcal{N}^\epsilon}\langle\psi, \nabla_{e_n}\psi\rangle \ \mathrm{dvol}_{\partial\mathcal{N}^\epsilon}$ is nonnegative. Finally,

$$-\frac{1}{4}\int_{\mathcal{N}^\epsilon}\langle \psi, [\gamma^\alpha, \gamma^\beta]F^V_{\alpha\beta}\psi\rangle \ \mathrm{dvol}_{\mathcal{N}^\epsilon} \ge -\,\mathrm{const.}\,\epsilon\int_{\mathcal{N}^\epsilon}|\psi|^2 \ \mathrm{dvol}_{\mathcal{N}^\epsilon}. \tag{2.34}$$

Using the uniform positivity of $\frac{R}{4} + f^2 - |\nabla f| - \epsilon f$ for small ϵ, we see that if ϵ is sufficiently small then we obtain a contradiction to (2.26). This proves the lemma. □

We will loosely refer to the index of $\mathcal{D}$, meaning the index of the restriction of $\mathcal{D}$ to $\mathrm{Dom}(\mathcal{D})^+$.

Lemma 2.35. *The index of $\mathcal{D}$ equals $\int_{\partial M}\widehat{A}(T\partial M)\wedge \mathrm{ch}\left(\beta\big|_{\partial M}\right)$.*

Proof. As we have local boundary conditions, the index only depends on the principal symbols of D^E and Π, up to homotopy. We can perform deformations of the geometry without changing the index, as long as we preserve ellipticity.

We first make the Riemannian metric on $\mathcal{N}^\epsilon$ a product near $\partial\mathcal{N}^\epsilon$. Let us then parametrize a tubular neighborhood of ∂M as $\partial M \times [0, \delta)$. Next, we deform the function f to zero. Deform the connection ∇^V on V to be a product near $\partial\mathcal{N}^\epsilon$. Construct a connection ∇^E on $E = V \oplus V$ so that on $\partial M \times [0, \delta/4]$, we have $\nabla^E = \begin{pmatrix} \nabla^V & 0 \\ 0 & \nabla^V\end{pmatrix}$, and on the complement of $\partial M \times [0, \delta/2]$, we have $\nabla^E = \begin{pmatrix} \nabla^V & 0 \\ 0 & \sigma \circ \nabla^V \circ \sigma^{-1} \end{pmatrix}$. If we were to perform a gauge transformation by σ^{-1} on E^- (the second V-factor in E) near $\partial_+ \mathcal{N}^\epsilon = \partial \mathcal{N}^\epsilon - \partial M$ then the result would be a product connection on E there.

We now use an argument as in [30, Proof of Theorem 1.1]. Let $D\mathcal{N}^\epsilon$ denote the double of $\mathcal{N}^\epsilon$. Let DE be the extension of E to $D\mathcal{N}^\epsilon$ so that when restricted to $\partial \mathcal{N}^\epsilon$, the involution on $D\mathcal{N}^\epsilon$ acts on $DE\Big|_{\partial \mathcal{N}^\epsilon} \cong E\Big|_{\partial \mathcal{N}^\epsilon}$ by switching the two V-factors. Then the boundary condition $\Pi\psi = \psi$ amounts to looking at $\mathbb{Z}_2$-invariant sections of $DS \otimes DE$. Hence the index of $\mathcal{D}$ is the same as the index of the Dirac-type operator acting on $\mathbb{Z}_2$-invariant sections of $DS \otimes DE$, which is same as the index of the orbifold Dirac-type operator on the $\mathbb{Z}_2$-quotient of $D\mathcal{N}^\epsilon$. Since $\partial \mathcal{N}^\epsilon$ is odd dimensional, it does not contribute to the orbifold index formula. Hence the index of $\mathcal{D}$ equals $\int_{\mathcal{N}^\epsilon} \widehat{A}(T\mathcal{N}^\epsilon) \wedge \mathrm{tr}_s \left(e^{\frac{i}{2\pi}(\nabla^E)^2} \right)$. By construction,

$$
\begin{aligned}
(2.36) \qquad \int_{\mathcal{N}^\epsilon} \widehat{A}(T\mathcal{N}^\epsilon) \wedge \mathrm{tr}_s \left(e^{\frac{i}{2\pi}(\nabla^E)^2} \right) &= \int_{\partial M \times [0,\delta]} \widehat{A}(T\mathcal{N}^\epsilon) \wedge \mathrm{tr}_s \left(e^{\frac{i}{2\pi}(\nabla^E)^2} \right) \\
&= \int_{\partial M} \widehat{A}(T\partial M) \wedge \int_{[0,\delta]} \mathrm{tr}_s \left(e^{\frac{i}{2\pi}(\nabla^E)^2} \right) \\
&= \int_{\partial M} \widehat{A}(T\partial M) \wedge \mathrm{ch}(V, \sigma)\big|_{\partial M} \\
&= \int_{\partial M} \widehat{A}(T\partial M) \wedge \mathrm{ch}\left(\beta\big|_{\partial M} \right).
\end{aligned}
$$

This proves the lemma. ☐

Combining Lemmas 2.19 and 2.35 proves the theorem when M is even dimensional. Suppose that M is odd dimensional. Consider $M' = M \times S^1$ and $K' = K \times S^1$. The theorem for M now follows from the theorem for M'. ☐

Corollary 2.37. *Suppose that Assumption 2.8 holds. Let Γ be a discrete group and let η be an element of $\mathrm{K}^*_{af}(B\Gamma)$. Given a continuous map $\nu : \mathcal{C} \to B\Gamma$, let $\nu\big|_{\partial M}$ be its restriction to ∂M. Then*

$$
(2.38) \qquad \int_{\partial M} \widehat{A}(T\partial M) \wedge \mathrm{ch}\left(\nu\big|_{\partial M}^* \eta \right) = 0.
$$

Remark 2.39. There is some overlap between Corollary 2.37 and [20, Theorem 6.12]. Namely, if M is a spin manifold-with-boundary with compact boundary then by taking $\mathcal{C}$ large enough, Corollary 2.37 gives obstructions to M having a complete Riemannian metric with positive scalar curvature and nonnegative mean curvature boundary. On the other hand, if there is such a metric then taking the double DM and smoothing the metric, we get a complete Riemannian metric with positive scalar curvature, to which [20, Theorem 6.12] gives some obstructions.

2.3. **End obstructions.** We show how Theorem 2.9 gives end obstructions to the existence of complete finite volume metrics with positive scalar curvature. We first prove a general result about the existence of hypersurfaces with nonnegative mean curvature, as seen from infinity, in a finite volume manifold. The proof was explained to me by Antoine Song.

Proposition 2.40. *Let M be a complete finite volume oriented Riemannian manifold with compact boundary, of dimension at most seven. Then there is an exhaustion of M by compact connected submanifolds-with-boundary Z so that ∂Z has nonnegative mean curvature, as seen from $M - Z$.*

Proof. The proof is along the lines of that of [37, Claim 2.4]. Fixing a basepoint $m_0 \in M$, slightly smoothing the distance function from m_0 on compact subsets, and applying the coarea formula, we see that there is an exhaustion of M by compact codimension-zero submanifolds $\{K_i\}_{i=1}^{\infty}$ with smooth boundary so that $\lim_{i \to \infty} \mathrm{vol}_{n-1}(\partial K_i) = 0$. The idea is now to move each ∂K_i in a 1-parameter family of hypersurfaces within K_i. Given i, consider a 1-parameter family $L = \{L_t\}_{t \in [0,1]}$ of n-dimensional integral currents in K_i so that

- $\partial L_0 = \partial K_i$,
- ∂L_t is a continuous function of t in the flat topology, and
- For all t, $\mathrm{vol}_{n-1}(\partial L_t) \le 2 \, \mathrm{vol}_{n-1}(\partial K_i)$.

Let Γ_i denote the collection of such 1-parameter families.

Fix a small ball B around m_0. We claim that for all large i, ∂L_1 is nonempty for any $L \in \Gamma_i$. Otherwise, the family $\{\partial L_t\}_{t \in [0,1]}$ would sweep out K_i. In particular, some ∂L_t would divide B into two parts of equal volume. However, for large i, this contradicts the assumption that $\mathrm{vol}_{n-1}(\partial L_t) \le 2 \, \mathrm{vol}_{n-1}(\partial K_i)$.

Now take a sequence $\{L^j\}_{j=1}^{\infty}$ in Γ_i so that $\mathrm{vol}_{n-1}(\partial L^j_1)$ approaches $\inf_{L \in \Gamma_i} \mathrm{vol}_{n-1}(\partial L_1)$. After passing to a subsequence, we can assume that $\lim_{j \to \infty} \partial L^j = W_i$ in the flat topology, for some $C^{1,1}$-regular hypersurface W_i. By construction, $\mathrm{vol}_{n-1} W_i \le 2 \, \mathrm{vol}_{n-1}(\partial K_i)$. Where W_i doesn't intersect ∂K_i, it will be a minimal hypersurface. Some of W_i may be hung up on ∂K_i but in any case, W_i will have nonnegative mean curvature in the weak sense as seen from $M - K_i$. Let $\widehat{Z}_i$ be the closure of the connected component of $M - W_i$ that contains B. By an argument as in the proof of the above claim, the $\widehat{Z}_i$'s exhaust M. Finally, by running the mean curvature flow for a short time, we can smooth W_i to a nearby smooth hypersurface H_i with nonnegative mean curvature. Letting Z_i denote the corresponding submanifold of M with boundary component H_i, we obtain the desired exhaustion of M. $\square$

Corollary 2.41. *Let M be a connected complete finite volume Riemannian spin manifold-with-boundary, with compact boundary, positive scalar curvature outside of a compact set, and dimension at most seven.*

Let Γ *be a discrete group and let* η *be an element of* $\mathrm{K}^*_{af}(B\Gamma)$*. Given a continuous map* $\nu : M \to B\Gamma$*, let* $\nu\big|_{\partial M}$ *be its restriction to* ∂M*. Then*

$$\int_{\partial M} \widehat{A}(T\partial M) \wedge \mathrm{ch}\left(\nu\big|_{\partial M}^{*}\eta\right) = 0. \tag{2.42}$$

Proof. We apply Proposition 2.40 to get Z, with $\partial M \subset Z$ and $\overline{M - Z}$ having positive scalar curvature. Let us write $\partial_+ Z$ for $\partial\overline{M - Z}$, so $\partial Z = \partial M \cup \partial_+ Z$. Applying Theorem 2.9 to a large compact submanifold $\mathcal{C}$ of $\overline{M - Z}$ containing $\partial_+ Z$ gives

$$\int_{\partial_+ Z} \widehat{A}(T\partial_+ Z) \wedge \mathrm{ch}\left(\nu\big|_{\partial_+ Z}^{*}\eta\right) = 0. \tag{2.43}$$

Deforming the metric on Z to be a product near the boundary gives

$$\begin{aligned} &\int_{\partial_+ Z} \widehat{A}(T\partial_+ Z) \wedge \mathrm{ch}\left(\nu\big|_{\partial_+ Z}^{*}\eta\right) - \int_{\partial M} \widehat{A}(T\partial M) \wedge \mathrm{ch}\left(\nu\big|_{\partial M}^{*}\eta\right) = \\ &\int_{Z} d\left(\widehat{A}(TZ) \wedge \mathrm{ch}\left(\nu\big|_{Z}^{*}\eta\right)\right) = 0. \end{aligned} \tag{2.44}$$

Equation (2.42) follows. □

Example 2.45. Let C be a compact connected spin manifold of dimension at most six. Let $\nu : C \to B\pi_1(C)$ be the classifying map for the universal cover of C, defined up to homotopy. If $[0, \infty) \times C$ has a complete finite volume Riemannian metric with positive scalar curvature then for any $\eta \in \mathrm{K}^*_{af}(B\pi_1(C))$, we have $\int_C \widehat{A}(TC) \wedge \mathrm{ch}\left(\nu^*\eta\right) = 0$.

For example, there is no finite volume complete Riemannian metric with positive scalar curvature on $[0, \infty) \times T^{n-1}$ if $n \leq 7$. Of course, such metrics exist if one removes the finite volume condition.

Remark 2.46. To see more clearly that Theorem 2.9 gives an end obstruction to positive scalar curvature, suppose that M is a complete noncompact finite volume Riemannian spin manifold having $\dim(M) \leq 7$, with positive scalar curvature outside of a compact subset. For notational simplicity, suppose that $\dim(M)$ is even; the case when $\dim(M)$ is odd is similar. Let $\{K_i\}$ be an exhaustion of M by compact submanifolds-with-boundary, with $K_i \subset \mathrm{int}(K_{i+1})$. Put $\mathcal{K}^{-1}_{af,i} = \lim_{\overleftarrow{j}} \mathrm{K}^{-1}_{af}(K_j - \mathrm{int}(K_i))$, an inverse limit, and then put $\mathcal{K}^{-1}_{af,\infty} = \lim_{\overrightarrow{i}} \mathcal{K}^{-1}_{af,i}$, a direct limit. Clearly $\mathcal{K}^{-1}_{af,\infty}$ only depends on the geometry of M at infinity. Given $\beta \in \mathcal{K}^{-1}_{af,\infty}$, we obtain an obstruction as follows. Choose a $\beta_i \in \mathcal{K}^{-1}_{af,i}$ representing β, for some i. Taking i large, we can assume that $M - K_i$ has positive scalar curvature. Take $\beta_{i,j} \in \mathrm{K}^{-1}_{af}(K_j - \mathrm{int}(K_i))$ in the inverse limit defining β_i. Spin cobordism implies that $\int_{\partial K_i} \widehat{A}(T\partial K_i) \wedge \mathrm{ch}\left(\beta_{i,j}\big|_{\partial K_i}\right)$ is independent of the choices made. Taking j sufficiently large, Theorem 2.9 implies that it vanishes.

3. Obstructions from almost flat relative classes

In this section we give obstructions to positive scalar curvature using almost flat classes in the relative setting. Subsection 3.1 recalls the notions of relative cohomology and relative

K-theory. In Subsection 3.2 we discuss the issue of basepoints for classifying maps in the relative setting, when the second space may not be path connected. Subsection 3.3 has the definition and basic properties of almost flat relative classes. In Subsection 3.4 we use almost flat relative classes to give obstructions for a compact manifold-with-boundary to admit a metric with positive scalar curvature and nonnegative mean curvature boundary. Finally, in Subsection 3.5 we give a local obstruction for a manifold to admit a metric with positive scalar curvature, using almost flat relative classes.

3.1. **Relative cohomology.** Let X be a compact topological space and let Y be a closed subset of X. Generators for the relative K-group $\mathrm{K}^0(X,Y)$ are pairs (E,σ) where $E = E^+ \oplus E^-$ is a $\mathbb{Z}_2$-graded complex vector bundle on X and $\sigma : E^+\Big|_Y \rightarrow E^-\Big|_Y$ is an isomorphism [24, Chapter 2, Section 2.29]. There is a Chern character $\mathrm{ch} : \mathrm{K}^0(X,Y) \rightarrow \mathrm{H}^{even}(X,Y;\mathbb{Q})$ [24, Chapter 5, Section 3.26]. We write $\mathrm{ch}(E,\sigma)$ for the Chern character of the pair (E,σ).

Now let X be a compact topological space and let Y be a compact topological space with a continuous map $\mu : Y \rightarrow X$. Generators for the relative K-group $\mathrm{K}^0(Y \rightarrow X)$ are pairs (E,σ) where $E = E^+ \oplus E^-$ is a $\mathbb{Z}_2$-graded complex vector bundle on X and $\sigma : \mu^* E^+ \rightarrow \mu^* E^-$ is an isomorphism.

Equivalently, in terms of mapping cylinders, let Z be the quotient of $X \cup ([0,1] \times Y)$ by the equivalence relation $\mu(y) \sim (\{0\} \times y)$. Put $Z' = \{1\} \times Y$, a subset of Z. Then $\mathrm{K}^*(Y \rightarrow X) = \mathrm{K}^*(Z, Z')$.

If X' and Y' are other such spaces, and we have maps $X \rightarrow X'$ and $Y \rightarrow Y'$ so that the diagram

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & X' \end{array} \tag{3.1}$$

commutes, we will denote it by a map $\nu : (X,Y) \rightarrow (X',Y')$. There is a pullback $\nu^* : \mathrm{K}^*(Y' \rightarrow X') \rightarrow \mathrm{K}^*(Y \rightarrow X)$.

As a variation, suppose that we additionally have a complex vector bundle V on Y, and an isomorphism $\sigma : \mu^* E^+ \oplus V \rightarrow \mu^* E^- \oplus V$. We can find some $N \in \mathbb{Z}^+$ and a vector bundle G on Y so that $V \oplus G$ can be identified with a trivial bundle $Y \times \mathbb{C}^N$. Put $\widehat{E}^{\pm} = E^{\pm} \oplus (X \times \mathbb{C}^N)$. Define an isomorphism $\widehat{\sigma} : \mu^* \widehat{E}^+ \rightarrow \mu^* \widehat{E}^-$ by $\widehat{\sigma} = \sigma \oplus \mathrm{Id}\Big|_G$. We put $[E,V,\sigma] = [\widehat{E},\widehat{\sigma}]$ as an element of $\mathrm{K}^0(Y \rightarrow X)$; it is independent of the choices made. We write $\mathrm{ch}(E,V,\sigma) = \mathrm{ch}(\widehat{E},\widehat{\sigma})$.

We define the relative cohomology groups $\mathrm{H}^*(Y \rightarrow X;\mathbb{Q})$ by the algebraic mapping cone construction. That is, the relative cochains are

$$C^k(Y \rightarrow X;\mathbb{Q}) = C^k(X;\mathbb{Q}) \oplus C^{k-1}(Y;\mathbb{Q}) \tag{3.2}$$

with differential $d(c^k, c^{k-1}) = (dc^k, \mu^* c^k - dc^{k-1})$. There are Chern characters $\mathrm{ch} : \mathrm{K}^0(Y \rightarrow X) \rightarrow \mathrm{H}^{even}(Y \rightarrow X;\mathbb{Q})$ and $\mathrm{ch} : \mathrm{K}^{-1}(Y \rightarrow X) \rightarrow \mathrm{H}^{odd}(Y \rightarrow X;\mathbb{Q})$.

Remark 3.3. If Y is a closed subset of X then $\mathrm{K}^*(Y \to X) = \mathrm{K}^*(X,Y)$ and $\mathrm{H}^*(Y \to X;\mathbb{Q}) = \mathrm{H}^*(X,Y;\mathbb{Q})$.

3.2. **Basepoints.** When talking about classifying maps for pairs (X,Y), where Y may be disconnected, there is an issue of how to deal with basepoints. We first describe classifying maps in terms of fundamental groupoids, where such issues do not arise. We then show how to translate this in terms of fundamental groups.

If $h : \Gamma' \to \Gamma$ is a homomorphism between discrete groups then we can talk about $\mathrm{K}^*(B\Gamma' \to B\Gamma)$ and $\mathrm{H}^*(B\Gamma' \to B\Gamma;\mathbb{Q})$. In the special case when h is injective, there are two different algebraic notions of the relative group cohomology [1, 38]. Our $\mathrm{H}^*(B\Gamma' \to B\Gamma;\mathbb{Q})$ corresponds to the one defined in [38].

If Γ is replaced by a topological groupoid $\mathcal{G}$ then there is again a classifying space $B\mathcal{G}$. Given a finite collection $\{\mathcal{G}_\alpha\}$ of topological groupoids with continuous homomorphisms $h_\alpha : \mathcal{G}_\alpha \to \mathcal{G}$ we can define $\mathrm{K}^*(\coprod_\alpha B\mathcal{G}_\alpha \to B\mathcal{G})$.

The groupoids $\mathcal{G}$ that are relevant for us are transitive étale groupoids. Here "transitive" means that the range and source maps $r, s : \mathcal{G}^{(1)} \to \mathcal{G}^{(0)}$ are such that $(r,s) : \mathcal{G}^{(1)} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ is surjective. Then $\mathcal{G}$ is Morita equivalent to a discrete group, namely the isotropy group $\mathcal{G}_p = r^{-1}(p) \cap s^{-1}(p)$ for any $p \in \mathcal{G}^{(0)}$.

We now describe the classifying map ν for pairs, arising from fundamental groupoids. Let X be a compact path connected topological space. Let $\mathcal{G}_X$ be the fundamental groupoid of X and let $B\mathcal{G}_X$ be its classifying space, as constructed in the sense of classifying spaces of topological categories. There is an inclusion $X \to B\mathcal{G}_X$. Let Y be a compact subset of X, with path components $\{Y_\alpha\}$. We obtain a commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & \coprod_\alpha B\mathcal{G}_{Y_\alpha} \\ \downarrow & & \downarrow \\ X & \longrightarrow & B\mathcal{G}_X. \end{array} \tag{3.4}$$

For brevity, we will write such a diagram as a map $(X,Y) \to (B\mathcal{G}_X, \coprod_\alpha B\mathcal{G}_{Y_\alpha})$ and will call it the classifying map for the pair (X,Y).

To write the classifying map in terms of groups, instead of groupoids, choose a basepoint $x_0 \in X$ and basepoints $y_\alpha \in Y_\alpha$. Let γ_α be a path from x_0 to y_α. This gives a homomorphism $\pi_1(Y_\alpha, y_\alpha) \to \pi_1(X, x_0)$. The path groupoid $\mathcal{G}_X$ is Morita-equivalent to $\pi_1(X, x_0)$, and similarly for $\mathcal{G}_{Y_\alpha}$. We obtain a diagram

$$\begin{array}{ccc} \coprod_\alpha B\mathcal{G}_{Y_\alpha} & \longrightarrow & \coprod_\alpha B\pi_1(Y_\alpha, y_\alpha) \\ \downarrow & & \downarrow \\ B\mathcal{G}_X & \longrightarrow & B\pi_1(X, x_0), \end{array} \tag{3.5}$$

depending on $\{\gamma_\alpha\}$, that commutes up to homotopy.

Putting (3.4) and (3.5) together, we obtain a diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \coprod_\alpha B\pi_1(Y_\alpha, y_\alpha) \arrow[d] \\ X \arrow[r] & B\pi_1(X, x_0), \end{tikzcd} \tag{3.6}$$

depending on $\{\gamma_\alpha\}$, that commutes up to homotopy. For brevity, we will write such a diagram as a map $(X, Y) \to (B\pi_1(X, x_0), \coprod_\alpha B\pi_1(Y_\alpha, y_\alpha))$. Different choices of $\{\gamma_\alpha\}$ will give different, but equivalent, groups and diagrams.

Let Γ be a discrete group and let $\{\Gamma'_\alpha\}$ be a finite collection of discrete groups with homomorphisms $h_\alpha : \Gamma'_\alpha \to \Gamma$. Given homomorphisms $\pi_1(X, x_0) \to \Gamma$ and $\pi_1(Y_\alpha, y_\alpha) \to \Gamma'_\alpha$ so that the diagrams

$$\begin{tikzcd} \pi_1(Y_\alpha, y_\alpha) \arrow[r] \arrow[d] & \Gamma'_\alpha \arrow[d] \\ \pi_1(X, x_0) \arrow[r] & \Gamma \end{tikzcd} \tag{3.7}$$

commute, we obtain a diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \coprod_\alpha B\Gamma'_\alpha \arrow[d] \\ X \arrow[r] & B\Gamma, \end{tikzcd} \tag{3.8}$$

which we will write as a map $\nu : (X, Y) \to (B\Gamma, B\Gamma')$. It is defined up to homotopy. Note that the maps $B\Gamma'_\alpha \to B\Gamma$ need not be inclusions, so ν is not a map of pairs in the conventional sense. Rather, it is to be interpreted as the diagram (3.8). There is still a pullback map $\nu^* : \mathrm{K}^*(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma) \to \mathrm{K}^*(Y \to X) \cong \mathrm{K}^*(X, Y)$.

3.3. **Almost flat relative classes.**

Definition 3.9. *Let $\mathcal{N}$ be a compact manifold-with-boundary. Let $\mathcal{N}'$ be a compact manifold-with-boundary with a smooth map $\mu : \mathcal{N}' \to \mathcal{N}$. Put Riemannian metrics on $\mathcal{N}$ and $\mathcal{N}'$. Given $\beta \in \mathrm{K}^0(\mathcal{N}' \to \mathcal{N})$, we say that β is almost flat if for every $\epsilon > 0$, we can find*

- *A $\mathbb{Z}_2$-graded Hermitian vector bundle E on $\mathcal{N}$ and an isometric isomorphism $\sigma : \mu^* E^+ \to \mu^* E^-$ with (E, σ) representing β, and*
- *A Hermitian connection ∇^E on E whose curvature satisfies $\|F^\pm\| \leq \epsilon$, and also $\|(\mu^* \nabla^E)\sigma\| \leq \epsilon$.*

Let $\mathrm{K}^0_{af}(\mathcal{N}' \to \mathcal{N})$ denote the almost flat classes. The above definition of an almost flat relative class is essentially the same as that in [26, Definition 3.17]. The definition in [26] is effectively when $\mathcal{N}'$ is a submanifold of $\mathcal{N}$.

Definition 3.10. *Let Γ be a discrete group and let $\{\Gamma'_\alpha\}$ be a finite collection of discrete groups, with homomorphisms $h_\alpha : \Gamma'_\alpha \to \Gamma$. Given $\eta \in \mathrm{K}^0(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma)$, we say* *that η is almost flat if for every choice of compact manifold-with-boundary $\mathcal{N}$, compact manifolds-with-boundary $\{\mathcal{N}'_\alpha\}$, smooth maps $\mu_\alpha : \mathcal{N}'_\alpha \to \mathcal{N}$ and continuous map of pairs $\nu : (\mathcal{N}, \mathcal{N}') \to (B\Gamma, B\Gamma')$, we have $\nu^*\eta \in \mathrm{K}^0_{af}(\coprod_\alpha \mathcal{N}'_\alpha \to \mathcal{N})$.*

Let $\mathrm{K}^0_{af}(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma)$ denote the almost flat classes. Note that if $B\Gamma$ and $\{B\Gamma'_\alpha\}$ are smooth compact manifolds then a class $\eta \in \mathrm{K}^0(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma)$ is almost flat in the sense of Definition 3.10 if and only if it is almost flat in the sense of Definition 3.9.

For the odd case, define the almost flat elements $\mathrm{K}^{-1}_{af}(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma)$ to be the elements of $\mathrm{K}^{-1}(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma)$ whose product with $[B\mathbb{Z}] \in \mathrm{K}^{-1}(B\mathbb{Z})$ lies in $\mathrm{K}^0_{af}(\coprod_\alpha B(\Gamma'_\alpha \times \mathbb{Z}) \to B(\Gamma \times \mathbb{Z}))$.

Example 3.11. Let Γ be a discrete group and let $\{\Gamma'_\alpha\}$ be a finite collection of trivial groups. The exact sequence in K-theory implies that

$$\mathrm{K}^0(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma) = \mathrm{Ker}\left(\mathrm{K}^0(B\Gamma) \to \oplus_\alpha \mathrm{K}^0(B\Gamma'_\alpha)\right). \tag{3.12}$$

Suppose that $B\Gamma$ can be represented by a compact manifold. Then $B\Gamma'_\alpha$ maps to a point in $B\Gamma$. It follows that $\mathrm{K}^0_{af}(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma) = \mathrm{K}^0(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma)$.

Example 3.13. Suppose that $\Gamma = \{e\}$ and $\Gamma_1 = \mathbb{Z}$. Then $B\Gamma = \mathrm{pt}$ and $B\Gamma_1 = S^1$. From the exact sequence in K-theory, the boundary map $\mathrm{K}^{-1}(S^1) \to \mathrm{K}^0(S^1 \to \mathrm{pt})$ is an isomorphism, so $\mathrm{K}^0(S^1 \to \mathrm{pt}) \cong \mathbb{Z}$. Let's see what happens if we take $\mathcal{N} = \mathrm{pt}$ and $\mathcal{N}'_1 = S^1$. The vector bundle E is just a $\mathbb{Z}_2$-graded inner product space on pt and ∇^E is trivial. The pullback $\mu_1^* E$ is a trivial $\mathbb{Z}_2$-graded Hermitian vector bundle on S^1 and $\mu_1^*\nabla^E$ is the trivial connection. If $\phi_1 : \mu_1^* E^+ \to \mu_1^* E^-$ is an isometric isomorphism then we can identify ϕ_1 with a map $\Phi : S^1 \to U(N)$ satisfying $\|\Phi^{-1}\Phi'\| \le \epsilon$. If ϵ is small, independent of N, then this forces $\det \Phi$ to have vanishing winding number (one can reduce to when Φ takes value in $U(1)^N$), so $\mathrm{K}^0_{af}(S^1 \to \mathrm{pt}) = 0$.

Remark 3.14. Example 3.13 shows that if $\Gamma' \to \Gamma$ is not injective then we cannot expect that $\mathrm{K}^*_{af}(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma) \otimes \mathbb{Q} = \mathrm{K}^*(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma) \otimes \mathbb{Q}$. Put $\widetilde{\Gamma}'_\alpha = \Gamma'_\alpha / \mathrm{Ker}(h_\alpha)$. The induced homomorphism $\widetilde{h}_\alpha : \widetilde{\Gamma}'_\alpha \to \Gamma$ is injective. As there is a map $\mathrm{K}^*_{af}(\coprod_\alpha B\widetilde{\Gamma}'_\alpha \to B\Gamma) \to \mathrm{K}^*_{af}(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma)$, by switching from Γ' to $\widetilde{\Gamma}'$ we can effectively just consider almost flat relative K-theory classes that come from h_α being injective. It seems conceivable that the map $\mathrm{K}^*_{af}(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma) \otimes \mathbb{Q} \to \mathrm{K}^*(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma) \otimes \mathbb{Q}$ is an isomorphism if the h_α's are injective, at least when $B\Gamma$ and the $B\Gamma'_\alpha$'s are finite CW-complexes.

3.4. **Manifolds with boundary.**

Theorem 3.15. *Let M be a compact Riemannian spin manifold-with-boundary. If M has positive scalar curvature and ∂M has nonnegative mean curvature then for all $\beta \in \mathrm{K}^*_{af}(M, \partial M)$, we have*

$$\int_M \widehat{A}(TM) \wedge \mathrm{ch}(\beta) = 0. \tag{3.16}$$

Proof. Suppose first that M is even dimensional. Let S denote the $\mathbb{Z}_2$-graded spinor bundle on M. As in Definition 3.9, let E be a $\mathbb{Z}_2$-graded Hermitian vector bundle on M, with an isometric isomorphism $\sigma : E^+\big|_{\partial M} \to E^-\big|_{\partial M}$, so that (E, σ) represents β. Given $\epsilon > 0$, let ∇^E be a Hermitian connection on E so that $\|F^{\pm}\| \le \epsilon$ and $\|\nabla^E \sigma\| \le \epsilon$. Give $S \otimes E$ the total $\mathbb{Z}_2$-grading. Let D^E be the Dirac-type operator on $C^\infty(M; S \otimes E)$. Then

$$(D^E)^2 = \nabla^* \nabla + \frac{1}{4} R - \frac{1}{4} \left[\gamma^\mu, \gamma^\nu \right] F^E_{\mu\nu}. \tag{3.17}$$

Define a projection Π on $C^\infty \left(\partial M; (S \otimes E)\big|_{\partial M} \right)$ by

$$\Pi = \begin{pmatrix} 0 & \sqrt{-1} \epsilon_S \gamma^n \sigma^{-1} \\ \sqrt{-1} \epsilon_S \gamma^n \sigma & 0 \end{pmatrix}. \tag{3.18}$$

Let $\mathcal{D}$ be the self-adjoint Dirac-type operator, densely defined on $L^2(M; S \otimes E)$, with boundary condition $(\Pi - I)\psi\big|_{\partial M} = 0$. As in the proof of Lemma 2.19, if ϵ is small enough then $\mathrm{Ker}(\mathcal{D}) = 0$.

To compute the index of $\mathcal{D}$, we use a doubling argument as in the proof of Lemma 2.35. Because of the local boundary conditions, the index only depends on the principal symbols of D^E and Π, up to homotopy. We deform the Riemannian metric and the Hermitian connection ∇^E so that they are products near ∂M, and we can also deform so that $\nabla^{E^-}\big|_{\partial M} = \sigma \circ \nabla^{E^+}\big|_{\partial M} \circ \sigma^{-1}$. The form $\mathrm{tr}_s\, e^{\frac{i}{2\pi} F^E}$ vanishes in a neighborhood of ∂M and represents $\mathrm{ch}(\beta)$.

We pass to the double DM, whose $\mathbb{Z}_2$-involution has fixed point set ∂M. We can extend $S \otimes E$ to DM so that the generator of the $\mathbb{Z}_2$-involution acts on $(S \otimes E)\big|_{\partial M}$ by $i \epsilon_S \gamma^n \otimes \begin{pmatrix} 0 & \phi^{-1} \\ \phi & 0 \end{pmatrix}$. The index of $\mathcal{D}$ is the same as the $\mathbb{Z}_2$-invariant index on DM, which is the same as the orbifold index on the orbifold $DM/\mathbb{Z}_2$. From the orbifold index theorem, the index equals $\int_M \widehat{A}(TM) \wedge \mathrm{tr}_s\, e^{\frac{i}{2\pi} F^E} = \int_M \widehat{A}(TM) \wedge \mathrm{ch}(\beta)$.

This proves the theorem in the even dimensional case. In the odd dimensional case, we take a product with S^1 to reduce to the even dimensional case. □

Corollary 3.19. *Let M be a compact Riemannian spin manifold-with-boundary. Let $\{\partial_\alpha M\}$ be the boundary components. Suppose that M has positive scalar curvature and ∂M has nonnegative mean curvature.*

Let Γ be a discrete group and let $\{\Gamma'_\alpha\}$ be a finite collection of discrete groups, with homomorphisms $h_\alpha : \Gamma'_\alpha \to \Gamma$. Given $\eta \in \mathrm{K}^*_{af} \left(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma \right)$ and a continuous map of pairs $\nu : (M, \partial M) \to (B\Gamma, B\Gamma')$, we have

$$\int_M \widehat{A}(TM) \wedge \mathrm{ch}(\nu^* \eta) = 0. \tag{3.20}$$

Corollary 3.21. *Let M be a compact Riemannian spin manifold-with-boundary. Let $\{\partial_\alpha M\}$ be the boundary components. Suppose that M has positive scalar curvature and ∂M has nonnegative mean curvature.*

Put $\Gamma = \pi_1(M)$ and $\Gamma'_\alpha = \pi_1(\partial_\alpha M)$, where basepoints are handled as in Section 3.2. Put $\widetilde{\Gamma}'_\alpha = \Gamma'_\alpha / \operatorname{Ker}(\Gamma'_\alpha \to \Gamma)$. Let ν be the composition $(M, \partial M) \to (B\Gamma, B\Gamma') \to (B\Gamma, B\widetilde{\Gamma}')$. Suppose that $\mathrm{K}^*_{af}\left(\coprod_\alpha B\widetilde{\Gamma}'_\alpha \to B\Gamma\right) \otimes \mathbb{Q} = \mathrm{K}^*\left(\coprod_\alpha B\widetilde{\Gamma}'_\alpha \to B\Gamma\right) \otimes \mathbb{Q}$. Then $\nu_*[M, \partial M]$ vanishes in $\mathrm{H}_{\dim(M)}(B\Gamma, B\widetilde{\Gamma}'; \mathbb{Q})$.

Proof. Given Corollary 3.19, the proof is similar to that of Corollary 2.7. □

Remark 3.22. The motivation to pass from Γ' to $\widetilde{\Gamma}'$ comes from Remark 3.14.

Remark 3.23. To put Corollary 3.21 in perspective, we discuss the index theory obstructions to the existence of positive scalar curvature metrics on compact manifolds-with-boundary.

Let M be a compact connected spin manifold-with-boundary. For simplicity, we just discuss the case when ∂M is nonempty and connected; the issue of basepoints in the general case can be handled as in Section 3.2. Choose a basepoint $m_0 \in \partial M$, put $\Gamma = \pi_1(M, m_0)$ and put $\Gamma' = \pi_1(\partial M, m_0)$.

Suppose first that we are given a metric $g_{\partial M}$ on ∂M with $R_{\partial M} > 0$, and we want to know if we can extend it to a metric on M so that $R_M > 0$ and $H_{\partial M} \geq 0$. Choose an extension to an arbitrary metric g_M on M. Since $R_{\partial M} > 0$, the Dirac operator on M, with APS boundary conditions, has a well defined index in $\mathrm{K}_*(C^*_{max}\Gamma)$, the K-theory of the maximal group C^*-algebra. If g_M has $R_M > 0$ and $H_{\partial M} \geq 0$ then the index vanishes.

If we ask for the obstruction to finding *some* metric g_M on M with $R_M > 0$, $H_{\partial M} \geq 0$ and $R_{\partial M} > 0$, then we are effectively allowing ourselves to perturb a given boundary metric $g_{\partial M}$ with $R_{\partial M} > 0$ to another boundary metric with positive scalar curvature. This changes the index by an element of $\operatorname{Im}(\mathrm{K}_*(C^*_{max}\Gamma') \to \mathrm{K}_*(C^*_{max}\Gamma))$. Hence the obstruction lies in $\operatorname{Coker}(\mathrm{K}_*(C^*_{max}\Gamma') \to \mathrm{K}_*(C^*_{max}\Gamma))$.

There is a relative group C^*-algebra $C^*_{max}(\Gamma, \Gamma')$ that fits into a short exact sequence

$$0 \to SC^*_{max}\Gamma \to C^*_{max}(\Gamma, \Gamma') \to C^*_{max}\Gamma' \to 0. \tag{3.24}$$

From the long exact sequence of K-groups, we can also think of the obstruction as lying in $\operatorname{Ker}(\mathrm{K}_{*-1}(C^*_{max}(\Gamma, \Gamma')) \to \mathrm{K}_{*-1}(C^*_{max}\Gamma'))$. This obstruction was constructed, as an element of $\mathrm{K}_{*-1}(C^*_{max}(\Gamma, \Gamma'))$, in [10]. The equivalence with the above use of the APS boundary conditions was shown in [34]. The fact that the element maps to zero in $\mathrm{K}_{*-1}(C^*_{max}\Gamma')$ is a reflection of the assumption that $R_{\partial M} > 0$.

It is known that the existence of a metric g_M with $R_M > 0$, $H_{\partial M} \geq 0$ and $R_{\partial M} > 0$ is equivalent to the existence of such a metric which is a product near ∂M [3]. In fact, the papers [10, 34] consider metrics that are products near ∂M.

We now remove the assumption that $R_{\partial M} > 0$. One obstruction to having $R_M > 0$ and $H_{\partial M} \geq 0$ comes from Theorem 3.15. Another approach is to take the double DM and smooth the resulting metric to get a $\mathbb{Z}_2$-invariant Riemannian metric on DM with $R_{DM} > 0$ [33]. Now $\pi_1(DM, m_0)$ is the amalgamated free product $\pi_1(M, m_0) \star_{\pi_1(\partial M, m_0)} \pi_1(M, m_0)$. Since the homomorphism $\pi_1(\partial M, m_0) \to \pi_1(M, m_0)$ may not be injective, put $\Gamma = \pi_1(M, m_0)$ and $\widetilde{\Gamma}' = \pi_1(\partial M, m_0) / \operatorname{Ker}(\pi_1(\partial M, m_0) \to \pi_1(M, m_0))$. Then the induced homomorphism $\widetilde{\Gamma}' \to \Gamma$ is injective and $\pi_1(DM, m_0) = \Gamma \star_{\widetilde{\Gamma}'} \Gamma$.

There are $\mathbb{Z}_2$-equivariant maps $\mathrm{K}_*(DM) \to \mathrm{K}_*(B\Gamma \cup_{B\widetilde{\Gamma}'} B\Gamma)) \to \mathrm{K}_*(C^*_{max}(\Gamma \star_{\widetilde{\Gamma}'} \Gamma))$. Taking into account that the involution on DM is orientation reversing, we can pass to the $\mathbb{Z}_2$-quotient to get maps

$$\mathrm{K}_*(M, \partial M) \to \mathrm{K}_*(B\Gamma, B\widetilde{\Gamma}') \to \mathrm{K}^-_*(C^*_{max}(\Gamma \star_{\widetilde{\Gamma}'} \Gamma)), \tag{3.25}$$

where $\mathrm{K}^-_*(C^*_{max}(\Gamma \star_{\widetilde{\Gamma}'} \Gamma))$ denotes the part of $\mathrm{K}_*(C^*_{max}(\Gamma \star_{\widetilde{\Gamma}'} \Gamma))$ that is odd under involution. Under the curvature assumption, the strong Novikov conjecture for C^*_{max} (formulated $\mathbb{Z}_2$-equivariantly) implies that the image of the fundamental class $[M, \partial M] \in \mathrm{K}_*(M, \partial M)$ in $\mathrm{K}_*(B\Gamma, B\widetilde{\Gamma}') \otimes \mathbb{Q}$ vanishes; compare with Corollary 3.21. If $\widetilde{\Gamma}' = \Gamma$ then it is known that a metric with $R_M > 0$ and $H_{\partial M} \geq 0$ always exists [33, Theorem 2.2].

3.5. **Local obstruction using almost flat relative bundles.** In this subsection we use almost flat relative bundles to give localized obstructions to having positive scalar curvature. We make the following assumption.

Assumption 3.26. *Given $r_0, D > 0$, put $r_0' = \frac{1}{256} r_0^2 D^2$ and $D' = D + \frac{32}{r_0 D}$. Let M be a Riemannian spin manifold, possible incomplete. Let K be a compact submanifold-with-boundary of M. Suppose that*

- *The distance neighborhood $N_{D'}(K)$ lies in a compact submanifold-with-boundary $\mathcal{C}$ of M,*
- *$R > 0$ on K,*
- *$R \geq r_0$ on $N_D(K) - K$ and*
- *$R \geq -r_0'$ on $N_{D'}(K) - N_D(K)$.*

Theorem 3.27. *Suppose that Assumption 3.26 holds. We can identify $\mathrm{K}^*\left((\mathcal{C} - \mathrm{int}(K)) \to \mathcal{C}\right)$ with $\mathrm{K}^*\left(\mathcal{C}, \mathcal{C} - \mathrm{int}(K)\right)$. Then given $\beta \in \mathrm{K}^*_{af}\left(\mathcal{C}, \mathcal{C} - \mathrm{int}(K)\right)$, we have*

$$\int_{\mathcal{C}} \widehat{A}(TM) \wedge \mathrm{ch}(\beta) = 0. \tag{3.28}$$

Proof. Suppose first that M is even dimensional. Let $\epsilon > 0$ be a small parameter, which we will adjust. Let $\mathcal{N}^\epsilon$ and f be as in the proof of Theorem 2.9. (We now have $\partial M = \emptyset$.) As in Definition 3.9, let (E, σ) be a pair that represents $\beta\big|_{\mathcal{N}^\epsilon}$, where E is a $\mathbb{Z}_2$-graded Hermitian vector bundle on $\mathcal{N}^\epsilon$ and $\sigma : E^+\big|_{\mathcal{N}^\epsilon - \mathrm{int}(K)} \to E^-\big|_{\mathcal{N}^\epsilon - \mathrm{int}(K)}$ is an isometric isomorphism.

Put

$$A = \begin{pmatrix} \nabla^+ & f\sigma^{-1} \\ f\sigma & \nabla^- \end{pmatrix}, \tag{3.29}$$

a superconnection on E. Let D^E be the quantization of A, i.e.

$$D^E = \begin{pmatrix} D^+ & \epsilon_S f\sigma^{-1} \\ \epsilon_S f\sigma & D^- \end{pmatrix}, \tag{3.30}$$

where $D^{\pm}$ is the Dirac operator on $C^{\infty}(\mathcal{N}^{\epsilon}; S \otimes E^{\pm})$ and ϵ_S is the $\mathbb{Z}_2$-grading operator on spinors on $\mathcal{N}^{\epsilon}$. Then

$$(D^E)^2 = \begin{pmatrix} (D^+)^2 + f^2 & 0 \\ 0 & (D^-)^2 + f^2 \end{pmatrix} + \sqrt{-1}\epsilon_S c(df) \begin{pmatrix} 0 & \sigma^{-1} \\ \sigma & 0 \end{pmatrix} + \sqrt{-1}\epsilon_S f \begin{pmatrix} 0 & c(\nabla^+ \circ \sigma^{-1} - \sigma^{-1} \circ \nabla^-) \\ c(\nabla^- \circ \sigma - \sigma \circ \nabla^+) & 0 \end{pmatrix}. \tag{3.31}$$

We impose the boundary condition $\Pi\psi = \psi$ on $\partial\mathcal{N}^{\epsilon}$, as in the proof of Theorem 2.9. Let $\mathcal{D}$ denote the ensuing self-adjoint operator. As in Lemma 2.19, there is an $\epsilon_0 > 0$ so that if $\epsilon < \epsilon_0$, and in addition $\|F^{\pm}\| \leq \epsilon$ and $\|\nabla^E \sigma\| \leq \epsilon$, then the kernel of $\mathcal{D}$ vanishes.

To compute the index of $\mathcal{D}$, we follow the method of proof of Lemma 2.35. We deform the metric on $\mathcal{N}^{\epsilon}$ so that it is a product near the boundary and we deform f to zero. We deform ∇^+ to be a product near the boundary. We deform ∇^- to be $\sigma \circ \nabla^+ \circ \sigma^{-1}$ near the boundary. By the doubling argument, the index of $\mathcal{D}$ equals $\int_{\mathcal{N}^{\epsilon}} \widehat{A}(T\mathcal{N}^{\epsilon}) \wedge \mathrm{tr}_s \left(e^{\frac{i}{2\pi}(\nabla^E)^2} \right)$. The form $\mathrm{tr}_s \left(e^{\frac{i}{2\pi}(\nabla^E)^2} \right)$ vanishes near $\partial\mathcal{N}^{\epsilon}$ and represents $\mathrm{ch}\left(\beta\big|_{\mathcal{N}^{\epsilon}} \right)$. Extending it by zero to $\mathcal{C}$ gives a representative of $\mathrm{ch}(\beta)$; note that $\mathcal{N} - \mathcal{N}^{\epsilon}$ is contained in $\mathcal{N} - \mathrm{int}(K)$. Hence the index equals

$$\int_{\mathcal{N}^{\epsilon}} \widehat{A}(T\mathcal{N}^{\epsilon}) \wedge \mathrm{tr}_s \left(e^{\frac{i}{2\pi}(\nabla^E)^2} \right) = \int_{\mathcal{C}} \widehat{A}(TM) \wedge \mathrm{ch}(\beta). \tag{3.32}$$

This proves the theorem if M is even dimensional. If M is odd dimensional, consider $M' = M \times S^1$ and $K' = K \times S^1$. Then we reduce to the even dimensional case. □

Example 3.33. Let Y be a compact connected spin manifold of even dimension. Let F be a finite subset of Y. Let $M = Y \#_F Z$ be the result of taking the connect sum of Y with a possibly noncompact and possibly disconnected spin manifold Z, over the points of F. (If F is a point then this would be the usual connect sum.) We assume that the connect sum is taken within a small neighborhood U of F. Let $\tau : M \to Y$ be a map which is the identity on $Y - U \subset M$ and maps $M - (Y - U)$ to U, while sending the complement of a compact subset $K \subset M$ to F.

Given an element $\delta \in \mathrm{Ker}\left(\mathrm{K}^0(Y) \to \mathrm{K}^0(F) \right)$, suppose that δ lies in $\mathrm{K}^0_{af}(Y)$. Representing δ by a $\mathbb{Z}_2$-graded Hermitian vector bundle $\widehat{E}$ with connection $\nabla^{\widehat{E}}$, put $E = \tau^* \widehat{E}$ and $\nabla^E = \tau^* \nabla^{\widehat{E}}$. There is an isometric isomorphism $\sigma : E^+\big|_{M - \mathrm{int}(K)} \to E^-\big|_{M - \mathrm{int}(K)}$, pulled back from F, with $\nabla^E \sigma = 0$. Suppose that M has a Riemannian metric which satisfies Assumption 3.26. We have constructed an element of $\mathrm{K}^0_{af}\left(\mathcal{C}, \mathcal{C} - \mathrm{int}(K) \right)$ and Theorem 3.27 implies that $\int_Y \widehat{A}(TY) \wedge \mathrm{ch}(\delta) = 0$. If Y is odd dimensional and $\delta \in \mathrm{K}^{-1}_{af}(Y)$ then we can take the product with S^1 to obtain the same result.

In particular, if M is complete with positive scalar curvature then by choosing appropriate $\mathcal{C}$, we conclude that $\int_Y \widehat{A}(TY) \wedge \mathrm{ch}(\delta) = 0$. This can be compared with [39, Theorem 0.2].

Example 3.34. We show that Theorem 3.27, or more precisely its proof, gives a localized version of [20, Theorem 6.12]. Suppose first that $\dim(M)$ is even. With reference to Assumption 3.26, given a finite cover $p : \widehat{\mathcal{C}} \to \mathcal{C}$ of $\mathcal{C}$, put $\widehat{K} = p^{-1}(K)$. Suppose that for every $\epsilon > 0$, there is a finite cover $p : \widehat{\mathcal{C}} \to \mathcal{C}$ and a map $m : \mathrm{int}(\widehat{K}) \to S^n$ so that

- m sends the complement of some compact subset of $\mathrm{int}(\widehat{K})$ to a point,
- m is ϵ-contracting on 2-forms, and
- m has a nonzero degree d.

Extend m to be a point map on $\widehat{\mathcal{C}} - \mathrm{int}(\widehat{K})$. Let W be a $\mathbb{Z}_2$-graded Hermitian vector bundle on S^n with a fixed Hermitian connection, so that $\mathrm{ch}(W) = e[S^n] \in \mathrm{H}^n(S^n; \mathbb{Q})$ for some $e \in \mathbb{Q} - \{0\}$. Define a $\mathbb{Z}_2$-graded Hermitian vector bundle on $\mathcal{C}$ by $E = p_* m^* W$, i.e. for $x \in \mathcal{C}$, the fiber E_x is $\oplus_{y \in p^{-1}(x)} (m^*W)_y$. Then the curvature of the induced connection on E is $O(\epsilon)$ in magnitude, and it vanishes outside of $\mathrm{int}(K)$

Note that although the connection on E is flat on $\mathcal{C} - \mathrm{int}(K)$, it will generally have nontrivial holonomy there. As m sends $\widehat{\mathcal{C}} - \mathrm{int}(\widehat{K})$ to a point, and $W^+\big|_{\mathrm{pt}} \cong W^-\big|_{\mathrm{pt}}$, there is a parallel isometric isomorphism $\sigma : E^+\big|_{\mathcal{C}-\mathrm{int}(K)} \to E^-\big|_{\mathcal{C}-\mathrm{int}(K)}$. Then $\mathrm{ch}(E, \sigma) = de[\mathcal{C}, \mathcal{C} - \mathrm{int}(K)] \in \mathrm{H}^n(\mathcal{C}, \mathcal{C} - \mathrm{int}(K); \mathbb{Q})$.

If Assumption 3.26 holds then the proof of Theorem 3.27 implies that $\int_{\mathcal{C}} \widehat{A}(TM) \wedge \mathrm{ch}(E, \sigma) = 0$. However,

$$\int_{\mathcal{C}} \widehat{A}(TM) \wedge \mathrm{ch}(E, \sigma) = \int_{\mathcal{C}} de[\mathcal{C}, \mathcal{C} - \mathrm{int}(K)] = de \neq 0, \tag{3.35}$$

which is a contradiction. If $\dim(M)$ is odd then we can take the product with a circle to get the same conclusion.

Hence the scalar curvature assumption cannot be satisfied. This can be viewed as a localized version of [20, Theorem 6.12], with the difference that we only consider finite covers in defining Λ^2-enlargeability, as in [19].

Corollary 3.36. *Suppose that Assumption 3.26 holds. Let Γ be a discrete group and let $\{\Gamma'_\alpha\}$ be a finite collection of discrete groups, with homomorphisms $h_\alpha : \Gamma'_\alpha \to \Gamma$. Given $\eta \in \mathrm{K}^*_{af}\left(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma\right)$ and a continuous map of pairs $\nu : (\mathcal{C}, \mathcal{C} - \mathrm{int}(K)) \to (B\Gamma, B\Gamma')$, we have*

$$\int_{\mathcal{C}} \widehat{A}(TM) \wedge \mathrm{ch}(\nu^* \eta) = 0. \tag{3.37}$$

Example 3.38. Let Γ be a discrete group. Put $\Gamma'_1 = \Gamma'_2 = \Gamma$. The exact sequence in K-theory gives $\mathrm{K}^0(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma) \cong \mathrm{K}^{-1}(B\Gamma)$. Suppose that $\widehat{\eta} \in \mathrm{K}^{-1}(B\Gamma)$ is almost flat in the sense of Definition 2.5. Then the corresponding relative element $\eta \in \mathrm{K}^0(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma)$ is almost flat, which can be seen as follows. Let $\mathcal{N}$, $\mathcal{N}'_1$ and $\mathcal{N}'_2$ be compact manifolds-with-boundary, with smooth maps $\mu_\alpha : \mathcal{N}'_\alpha \to \mathcal{N}$. Let $\nu : (\mathcal{N}, \mathcal{N}') \to (B\Gamma, B\Gamma')$ be a continuous map. Let (V, σ) be as in Definitions 2.4 and 2.5. We rename σ as σ_{odd}. Put $E = V \oplus V$, with the obvious $\mathbb{Z}_2$-grading. Define $\sigma_1 : \mu_1^* E^+ \to \mu_1^* E^-$ to be the identity on $\mu_1^* V$ and define $\sigma : \mu_2^* E^+ \to \mu_2^* E^-$ to be $\mu_2^* \sigma_{odd}$. Then $(E, \{\sigma_\alpha\})$ satisfies the requirements of Definition 3.9. Multiplying by $\mathbb{Z}$, there is a similar construction of elements of $\mathrm{K}^{-1}_{af}(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma)$.

Now let M be a complete Riemannian manifold with positive scalar curvature. Let Z be a separating hypersurface. Given $\nu : M \to B\Gamma$, Theorem 3.27 implies that $\int_Z \widehat{A}(TZ) \wedge \mathrm{ch}(\nu\big|_Z^* \widehat{\eta}) = 0$. This can be compared with [7, Theorem 1.5], [11, Theorem 1.1] and [20, Corollary 6.8 and Theorem 6.12].

Example 3.39. As an extension of the previous example, let Γ'_1, Γ'_2 and Γ_3 be discrete groups, with injective homomorphisms $\Gamma_3 \to \Gamma'_1$ and $\Gamma_3 \to \Gamma'_2$. Put $\Gamma = \Gamma'_1 \star_{\Gamma_3} \Gamma'_2$, the amalgamated free product. There are injective homomorphisms $\Gamma'_1 \to \Gamma$ and $\Gamma'_2 \to \Gamma$. The exact sequence in K-theory gives $\mathrm{K}^*(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma) \cong \mathrm{K}^{*-1}(B\Gamma_3)$. Given an almost flat element $\widehat{\eta} \in \mathrm{K}^{*-1}(B\Gamma_3)$ in the sense of Definition 2.5, it is not immediate that the corresponding element $\eta \in \mathrm{K}^*(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma)$ is always almost flat, but suppose that it is. Then we can conclude the following. Let M be a complete Riemannian manifold with positive scalar curvature. Let Z be a separating connected π_1-injective hypersurface, dividing M into pieces M_1 and M_2, with $M_1 \cap M_2 = Z$. Put $\Gamma_3 = \pi_1(Z, z_0)$, $\Gamma'_1 = \pi_1(M_1, z_0)$ and $\Gamma'_2 = \pi_1(M_2, z_0)$. Let $\nu_Z : Z \to B\Gamma_3$ be the classifying map for the universal cover of Z. Theorem 3.27 implies that $\int_Z \widehat{A}(TZ) \wedge \mathrm{ch}(\nu_Z^* \widehat{\eta}) = 0$.

Corollary 3.40. *Suppose that we are given*

- *A compact spin manifold-with-boundary Y, with boundary components $\{Y'_\alpha\}$,*
- *Discrete groups Γ and $\{\Gamma'_\alpha\}$, with homomorphisms $h_\alpha : \Gamma'_\alpha \to \Gamma$, and*
- *A continuous map $\nu : (Y, \partial Y) \to (B\Gamma, B\Gamma')$.*

Suppose that the interior of Y has a complete Riemannian metric with positive scalar curvature. Then for any $\eta \in \mathrm{K}^*_{af}(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma)$, we have $\int_Y \widehat{A}(TY) \wedge \mathrm{ch}(\nu^*\eta) = 0$.

Proof. Taking K and $\mathcal{C}$ to be sufficiently large compact subsets of $\mathrm{int}(Y)$ that are diffeomorphic to Y, the corollary follows from Theorem 3.27. □

Corollary 3.41. *Let Y be a compact connected spin manifold-with-boundary, with boundary components $\{Y'_\alpha\}$. Put $\Gamma = \pi_1(Y)$ and $\Gamma'_\alpha = \pi_1(Y_\alpha)$, where basepoints are handled as in Section 3.2. Suppose that the interior of Y has a complete Riemannian metric with positive scalar curvature, and $\mathrm{K}^*_{af}(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma) \otimes \mathbb{Q} = \mathrm{K}^*(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma) \otimes \mathbb{Q}$. Let $\nu : (Y, \partial Y) \to (B\Gamma, B\Gamma')$ be the classifying map. Then $\nu_*[Y, \partial Y]$ vanishes in $\mathrm{H}_{\dim(Y)}(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma; \mathbb{Q})$.*

Proof. Given Corollary 3.40, the proof is similar to that of Corollary 2.7. □

4. Obstructions from almost flat stable relative bundles

In this section we give local obstructions to positive scalar curvature using almost flat stable relative bundles. We use it to give obstructions for a manifold to admit a complete finite volume metric with positive scalar curvature.

Definition 4.1. *Let $\mathcal{N}$ be a compact manifold-with-boundary. Let $\mathcal{N}'$ be a compact manifold-with-boundary with a smooth map $\mu : \mathcal{N}' \to \mathcal{N}$. Put Riemannian metrics on $\mathcal{N}$ and $\mathcal{N}'$. Given $\beta \in \mathrm{K}^0(\mathcal{N}' \to \mathcal{N})$, we say that β is an almost flat stable class if for every $\epsilon > 0$, we can find*

- *A $\mathbb{Z}_2$-graded Hermitian vector bundle E on $\mathcal{N}$ and a Hermitian vector bundle V on $\mathcal{N}'$,*
- *An isometric isomorphism $\sigma : \mu^* E^+ \oplus V \to \mu^* E^- \oplus V$ with (E, V, σ) representing β, and*
- *Hermitian connections ∇^E and ∇^V on E and V, respectively, whose curvatures satisfies $\| F^E \| \le \epsilon$ and $\| F^V \| \le \epsilon$, and also $\| \nabla \sigma \| \le \epsilon$.*

Let $\mathrm{K}^0_{af,st}(\mathcal{N}' \to \mathcal{N})$ denote the almost flat stable classes. The above definition of an almost flat stable class is essentially the same as that in [26, Definition 3.17]. The definition in [26] is effectively when μ is an embedding.

Definition 4.2. *Let Γ be a discrete group and let $\{\Gamma'_\alpha\}$ be a finite collection of discrete groups, with homomorphisms $h_\alpha : \Gamma'_\alpha \to \Gamma$. Given $\eta \in \mathrm{K}^0(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma)$, we say that η is an almost flat stable class if for every compact manifold-with-boundary $\mathcal{N}$, compact manifolds-with-boundary $\{\mathcal{N}'_\alpha\}$, smooth maps $\mu_\alpha : \mathcal{N}'_\alpha \to \mathcal{N}'$ and continuous map of pairs $\nu : (\mathcal{N}, \mathcal{N}') \to (B\Gamma, B\Gamma')$, we have $\nu^* \eta \in \mathrm{K}^0_{af,st}(\coprod_\alpha \mathcal{N}'_\alpha \to \mathcal{N})$.*

Let $\mathrm{K}^0_{af,st}(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma)$ denote the almost flat stable classes. Note that if $B\Gamma$ and $\{B\Gamma'_\alpha\}$ are smooth manifolds then $\eta \in \mathrm{K}^0(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma)$ is an almost flat stable class in the sense of Definition 4.2 if and only if it is an almost flat stable class in the sense of Definition 4.1.

Define the almost flat stable elements $\mathrm{K}^{-1}_{af,st}(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma)$ to be the elements of $\mathrm{K}^{-1}(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma)$ whose product with $[B\mathbb{Z}] \in \mathrm{K}^{-1}(B\mathbb{Z})$ lies in $\mathrm{K}^0_{af,st}(\coprod_\alpha B(\Gamma'_\alpha \times \mathbb{Z}) \to B(\Gamma \times \mathbb{Z}))$.

It is conceivable that for all discrete groups Γ and $\{\Gamma'_\alpha\}$, and homomorphisms $h_\alpha : \Gamma'_\alpha \to \Gamma$, one has $\mathrm{K}^*_{af,st}(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma) \otimes \mathbb{Q} = \mathrm{K}^*(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma) \otimes \mathbb{Q}$, at least if $B\Gamma$ and the $B\Gamma'_\alpha$'s are finite CW-complexes.

Example 4.3. Let Γ be the trivial group and let Γ'_1 be any discrete group. The exact sequence in K-theory gives $\mathrm{K}^*(B\Gamma'_1 \to B\Gamma) \cong \mathrm{K}^{*-1}(B\Gamma'_1)/\mathrm{K}^{*-1}(\mathrm{pt})$. Suppose that $\widehat{\eta} \in \mathrm{K}^{-1}(B\Gamma'_1)$ is almost flat in the sense of Definition 2.5. Let $\mathcal{N}$ and $\mathcal{N}'_1$ be compact Riemannian manifolds-with-boundary, with a smooth map $\mu_1 : \mathcal{N}'_1 \to \mathcal{N}$. Let $\nu : (\mathcal{N}, \mathcal{N}'_1) \to (B\Gamma, B\Gamma'_1)$ be a continuous map. With reference to Definition 2.4, let $(V, \widehat{\sigma})$ be an almost flat pair that represents $(\nu\big|_{\mathcal{N}'_1})^* \widehat{\eta}$. With reference to Definition 4.1, let $E^\pm$ be trivial vector bundles on $\mathcal{N}$ with a product connection. Put $\sigma = \mathrm{Id} \oplus \widehat{\sigma}$. Then (E, V, σ) represents an almost flat stable class in $\mathrm{K}^0(\mathcal{N}'_1 \to \mathcal{N})$. Hence $\mathrm{K}^0_{af,st}(B\Gamma'_1 \to B\Gamma) = \mathrm{K}^0(B\Gamma'_1 \to B\Gamma)$. There is a similar statement for $\mathrm{K}^{-1}_{af,st}(B\Gamma'_1 \to B\Gamma)$. Compare with Example 3.13.

In this section we make the following assumption.

Assumption 4.4. *Given $r_0, D > 0$, put $r'_0 = \frac{1}{256} r_0^2 D^2$ and $D' = D + \frac{32}{r_0 D}$. Let M be a Riemannian spin manifold, possibly incomplete. Let K be a compact codimension-zero submanifold-with-boundary in M. Suppose that*

- *The distance neighborhood $N_{D'}(K)$ lies in a compact submanifold-with-boundary $\mathcal{C}$ of M,*

- *$R > 0$ on K,*
- *$R \geq r_0$ on $N_D(K) - K$,*
- *$R \geq -r_0'$ on $N_{D'}(K) - N_D(K)$, and*
- *∂K has nonnegative mean curvature as seen from $M - K$.*

Theorem 4.5. *Suppose that Assumption 4.4 holds. We can identify $\mathrm{K}^*\left((\mathcal{C} - \mathrm{int}(K)) \to \mathcal{C}\right)$ with $\mathrm{K}^*\left(\mathcal{C}, \mathcal{C} - \mathrm{int}(K)\right)$. Then given $\beta \in \mathrm{K}^*_{af,st}\left(\mathcal{C}, \mathcal{C} - \mathrm{int}(K)\right)$, we have*

$$\int_{\mathcal{C}} \widehat{A}(TM) \wedge \mathrm{ch}(\beta) = 0. \tag{4.6}$$

Proof. Suppose first that M is even dimensional. Let $\epsilon > 0$ be a small parameter, which we will adjust. Let $\mathcal{N}^\epsilon$ and f be as in the proof of Theorem 2.9. As in Definition 4.1, let (E, V, σ) be a triple that represents β, where E is a $\mathbb{Z}_2$-graded Hermitian vector bundle on $\mathcal{N}^\epsilon$, V is a Hermitian vector bundle on $\mathcal{N}^\epsilon - \mathrm{int}(K)$ and $\sigma : E^+\big|_{\mathcal{N}^\epsilon - \mathrm{int}(K)} \oplus V \to E^-\big|_{\mathcal{N}^\epsilon - \mathrm{int}(K)} \oplus V$ is an isometric isomorphism. Put $W = V \oplus V$, a $\mathbb{Z}_2$-graded vector bundle, and put $\nabla^W = \nabla^V \oplus \nabla^V$. On the complement of $\mathrm{int}(K)$, put $Z = E \oplus W$, with Hermitian connection $\nabla^\pm = \nabla^{E_\pm} \oplus \nabla^{W_\pm}$.

Define Π' as in the top of (2.18), acting on sections of $(S \otimes W)\Big|_{\partial K}$. Define Π as in the bottom of (2.18), acting on sections of $(S \otimes Z)\Big|_{\partial \mathcal{N}^\epsilon}$. We will define a Dirac-type operator D acting on elements

$$(\psi, \psi') \in C^\infty(\mathcal{N}^\epsilon; S \otimes E) \oplus C^\infty(\mathcal{N}^\epsilon - \mathrm{int}(K); S \otimes W) \tag{4.7}$$

satisfying the boundary conditions $(\Pi' - I)\psi'\Big|_{\partial K} = 0$ and $(\Pi - I)(\psi \oplus \psi')\Big|_{\partial \mathcal{N}^\epsilon} = 0$. On K, we take D to be the standard Dirac operator on $C^\infty(K; S \otimes E)$. On $\mathcal{N}^\epsilon - \mathrm{int}(K)$, where D acts on sections of $S \otimes Z$, we take D to be

$$D = \begin{pmatrix} D^+ & \epsilon_S f \sigma^{-1} \\ \epsilon_S f \sigma & D^- \end{pmatrix}. \tag{4.8}$$

Here $D^\pm$ is the Dirac operator on sections of $S \otimes Z^\pm$ and ϵ_S is the $\mathbb{Z}_2$-grading operator on spinors on M. Then on $\mathcal{N}^\epsilon - \mathrm{int}(K)$,

$$\begin{aligned} D^2 = &\begin{pmatrix} (D^+)^2 + f^2 & 0 \\ 0 & (D^-)^2 + f^2 \end{pmatrix} + \sqrt{-1}\epsilon_S c(df) \begin{pmatrix} 0 & \sigma^{-1} \\ \sigma & 0 \end{pmatrix} + \\ &\sqrt{-1}\epsilon_S f \begin{pmatrix} 0 & c(\nabla^+ \circ \sigma^{-1} - \sigma^{-1} \circ \nabla^-) \\ c(\nabla^- \circ \sigma - \sigma \circ \nabla^+) & 0 \end{pmatrix}. \end{aligned} \tag{4.9}$$

Let $\mathcal{D}$ denote the ensuing self-adjoint operator with dense domain (4.7). As in Lemma 2.19, there is an $\epsilon_0 > 0$ so that if $\epsilon < \epsilon_0$, and in addition $\|F^\pm\| \leq \epsilon$ and $\|\nabla \sigma\| \leq \epsilon$, then the kernel of $\mathcal{D}$ vanishes.

Applying the doubling methods in the proofs of Lemma 2.35 and Theorem 3.27 gives that the index of $\mathcal{D}$ equals $\int_{\mathcal{C}} \widehat{A}(TM) \wedge \mathrm{ch}(\beta)$.

If M is odd dimensional, consider $M' = M \times S^1$ and $K' = K \times S^1$. Then we reduce to the even dimensional case. $\square$

Corollary 4.10. *Suppose that Assumption 4.4 holds. Let* Γ *be a discrete group and let* $\{\Gamma'_\alpha\}$ *be a finite collection of discrete groups, with homomorphisms* $h_\alpha : \Gamma'_\alpha \to \Gamma$*. Given* $\eta \in \mathrm{K}^*_{af,st}\left(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma\right)$ *and a continuous map of pairs* $\nu : (\mathcal{C}, \mathcal{C} - \mathrm{int}(K)) \to (B\Gamma, B\Gamma')$*, we have*

$$\int_{\mathcal{C}} \widehat{A}(TM) \wedge \mathrm{ch}(\nu^* \eta) = 0. \tag{4.11}$$

Corollary 4.12. *Suppose that Assumption 4.4 holds, with* $\mathcal{C}$ *connected. Let* $\{(\mathcal{C} - \mathrm{int}(K))_\alpha\}$ *be the connected components of* $\mathcal{C} - \mathrm{int}(K)$*. Put* $\Gamma = \pi_1(\mathcal{C})$ *and* $\Gamma'_\alpha = \pi_1\left((\mathcal{C} - \mathrm{int}(K))_\alpha\right)$*, where basepoints are handled as in Section 3.2. Let* $\nu : (\mathcal{C}, \mathcal{C} - \mathrm{int}(K)) \to (B\Gamma, B\Gamma')$ *be the classifying map.*

If $\mathrm{K}^*_{af,st}\left(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma\right) \otimes \mathbb{Q} = \mathrm{K}^*\left(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma\right) \otimes \mathbb{Q}$ *then* $\nu_*[\mathcal{C}, \mathcal{C} - \mathrm{int}(K)]$ *vanishes in* $\mathrm{H}_{\dim(M)}\left(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma\right) \otimes \mathbb{Q}$*.*

Proof. Given Corollary 4.10, the proof is similar to that of Corollary 2.7. □

Corollary 4.13. *Let* M *be a complete finite volume Riemannian spin manifold with positive scalar curvature and* $\dim(M) \leq 7$*. Then there is an exhaustion* $K_1 \subset K_2 \subset \ldots$ *of* M *by compact submanifolds so that for each* $j > i \geq 1$ *and each* $\beta \in \mathrm{K}^*_{af,st}(K_j, K_j - \mathrm{int}(K_i))$*, we have* $\int_{K_j} \widehat{A}(TM) \wedge \mathrm{ch}(\beta) = 0$*.*

Proof. Proposition 2.40 gives a sequence $\{K_i\}$ of compact submanifolds that exhaust M so that ∂K_i has nonnegative mean curvature as seen from $M - K_i$. Given i, if j is large enough then we can take $\mathcal{C} = K_j$ and satisfy Assumption 4.4. After passing to a subsequence of the K_i's, the corollary follows from Theorem 4.5. □

Corollary 4.14. *Suppose that we are given*

- *A compact spin manifold-with-boundary* Y*, with boundary components* $\{Y'_\alpha\}$*,*
- *Discrete groups* Γ *and* $\{\Gamma'_\alpha\}$*, with homomorphisms* $h_\alpha : \Gamma'_\alpha \to \Gamma$*, and*
- *A continuous map* $\nu : (Y, \partial Y) \to (B\Gamma, B\Gamma')$*.*

Suppose that $\dim(Y) \leq 7$ *and the interior of* Y *has a complete finite volume Riemannian metric with positive scalar curvature. Then for any* $\eta \in \mathrm{K}^*_{af,st}(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma)$*, we have* $\int_Y \widehat{A}(TY) \wedge \mathrm{ch}(\nu^* \eta) = 0$*.*

Proof. For small $\epsilon > 0$, let $T_\epsilon(\partial Y)$ be the ϵ-neighborhood of ∂Y in Y. We can deform ν to a map $\widehat{\nu} : (Y, T_\epsilon(\partial Y)) \to (B\Gamma, B\Gamma')$. Let $\{K_i\}$ be as in Corollary 4.13. If i is sufficiently large then for $j > i$, the class $\mathrm{ch}\left(\widehat{\nu}\Big|^*_{(K_j, K_j - \mathrm{int}(K_i))} \eta\right)$ can be represented by a differential form with support in K_i that extends by zero to a representative of $\mathrm{ch}(\nu^* \eta) \in \mathrm{H}^*_c(\mathrm{int}(Y); \mathbb{Q}) \cong \mathrm{H}^*(Y, \partial Y; \mathbb{Q})$. It follows from Corollary 4.13 that $\int_Y \widehat{A}(TY) \wedge \mathrm{ch}(\nu^* \eta) = \int_{K_j} \widehat{A}(TY) \wedge \mathrm{ch}(\widehat{\nu}^* \eta) = 0$. The corollary follows. □

Corollary 4.15. *Let* Y *be a compact connected spin manifold-with-boundary, with boundary components* $\{Y'_\alpha\}$*. Put* $\Gamma = \pi_1(Y)$ *and* $\Gamma'_\alpha = \pi_1(Y_\alpha)$*, where basepoints are handled as in Section 3.2. Suppose that*

- $\mathrm{int}(Y)$ *has a complete finite volume metric with positive scalar curvature,*
- $\dim(Y) \le 7$ *and*
- $\mathrm{K}^*_{af,st}(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma) \otimes \mathbb{Q} = \mathrm{K}^*(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma) \otimes \mathbb{Q}$.

Then $\nu_*[Y, \partial Y]$ *vanishes in* $\mathrm{H}_{\dim(Y)}(\coprod_\alpha B\Gamma'_\alpha \to B\Gamma; \mathbb{Q})$.

Proof. Given Corollary 4.14, the proof is similar to that of Corollary 2.7. □

Example 4.16. Suppose that ∂Y is connected. If $\pi_1(Y) = \{e\}$ then there is some intersection with Example 2.45. If $\pi_1(\partial Y) = \{e\}$ then there is some intersection with Example 3.33, when the manifold Z there is a copy of $\mathbb{R}^n$. If $\pi_1(Y)$ and $\pi_1(\partial Y)$ are both nontrivial free abelian groups then Corollary 4.15 applies, which goes beyond the previous examples.

Appendix A

In this appendix we discuss the relationship between simplicial volume and the index theoretic results in the body of the paper. In Subsection A.1 we review simplicial volume and l_1-homology. In Subsection A.2 we consider the simplicial volume for closed Riemannian manifolds with almost nonnegative scalar curvature, in two cases. The first case is when there is an upper diameter bound and a lower bound on the curvature operator. The second case is when the volume is normalized to one and there is a double sided curvature bound. Our results in the second case are inconclusive.

A.1. **Review of simplicial volume.** Let M be a closed compact connected oriented n-manifold. The simplicial volume of M [15] is

$$\| M \| = \inf \left\{ \sum_i |a_i| \ : \ \left[\sum_i a_i c_i \right] = [M] \right\}, \tag{A.1}$$

where $[M] \in \mathrm{H}_n(M; \mathbb{R})$ is the fundamental class in singular homology, each c_i is a singular simplex $c_i : \triangle^n \to M$ and the sum is finite. One knows that $\| M \|$ is a homotopy invariant and just depends on the image $\nu_*[M] \in \mathrm{H}_n(B\pi_1(M); \mathbb{R})$ of $[M]$ under the classifying map $\nu : M \to B\pi_1(M)$.

In what follows we will refer to countable CW-complexes, although the statements will apply to spaces that are homotopy equivalent to countable CW-complexes. If X is a countable CW-complex then the l_1-chains of X are defined by

$$C^{l_1}_k(X) = \left\{ \sum_i a_i c_i : \sum_i |a_i| < \infty \right\}, \tag{A.2}$$

where each c_i is a singular k-simplex of X. The usual boundary operator gives a map $\partial : C^{l_1}_k(X) \to C^{l_1}_{k-1}(X)$. The corresponding l_1-homology groups are $\mathrm{H}^{l_1}_*(X) = \mathrm{Ker}(\partial)/\mathrm{Im}(\partial)$. They acquire quotient seminorms $\| \cdot \|_{l_1}$. There is a natural map $\mathrm{H}_*(X; \mathbb{Q}) \to \mathrm{H}^{l_1}_*(X)$. If M is a compact oriented manifold then $\| M \| = \| [M] \|_{l^1}$. Despite the name, l_1-homology is not a homology theory, in that the excision axiom is not satisfied, which causes complications.

If X is connected and $\nu : X \to B\pi_1(X, x_0)$ is the canonical map, defined up to homotopy, then $\nu_* : \mathrm{H}^{l_1}_*(X) \to \mathrm{H}^{l_1}_*(B\pi_1(X, x_0))$ is an isometric isomorphism [27].

If X is a countable CW-complex and Y is a subcomplex of X then there is an inclusion of $C_k^{l_1}(Y)$ into $C_k^{l_1}(X)$ and we can define the relative l_1-chains by $C_k^{l_1}(X,Y) = C_k^{l_1}(X)/C_k^{l_1}(Y)$. There is an induced seminorm on $C_k^{l_1}(X,Y)$ and also an induced seminorm on the homology groups $\mathrm{H}_*^{l_1}(X,Y)$ of the corresponding chain complex.

More generally, let X be a countable CW-complex and let Y be a countable CW-complexes equipped with a cellular map $\mu : Y \to X$. We define the relative l_1-homology groups $\mathrm{H}_*^{l_1}(Y \to X)$ by the algebraic mapping cone construction. That is, the relative chains are

$$C_k^{l_1}(Y \to X) = C_k^{l_1}(X) \oplus C_{k-1}^{l_1}(Y) \tag{A.3}$$

with boundary operator $\partial(c_k, c_{k-1}) = (\partial c_k + \mu_* c_{k-1}, -\partial c_{k-1})$.

If Y is a subcomplex of X then the chain map $C_k^{l_1}(Y \to X) \to C_k^{l_1}(X,Y)$, given by $(c_k, c_{k-1}) \to c_k \mod C_k^{l_1}(Y)$, induces an isomorphism $\mathrm{H}_*^{l_1}(Y \to X) \cong \mathrm{H}_*^{l_1}(X,Y)$ which is in fact isometric [6, Lemma 5.1] although we won't need this.

Proposition A.4. *Let X be a compact path connected topological space. Let Y be a compact subset of X, with path components $\{Y_\alpha\}$. Suppose that (X,Y) is homotopy equivalent to a pair of countable CW-complexes. Then*

$$\mathrm{H}_*^{l_1}(X,Y) \cong \mathrm{H}_*^{l_1}\left(\coprod_\alpha B\pi_1(Y_\alpha, y_\alpha) \to B\pi_1(X, x_0) \right), \tag{A.5}$$

an isomorphism of topological vector spaces.

Proof. We have short exact sequences

$$\mathrm{H}_*^{l_1}(Y) \to \mathrm{H}_*^{l_1}(X) \to \mathrm{H}_*^{l_1}(Y \to X) \to \mathrm{H}_{*-1}^{l_1}(Y) \to \mathrm{H}_{*-1}^{l_1}(X) \tag{A.6}$$

and

$$\begin{aligned} &\mathrm{H}_*^{l_1}\left(\coprod_\alpha B\pi_1(Y_\alpha, y_\alpha) \right) \to \mathrm{H}_*^{l_1}(B\pi_1(X, x_0)) \to \\ &\mathrm{H}_*^{l_1}\left(\coprod_\alpha B\pi_1(Y_\alpha, y_\alpha) \to B\pi_1(X, x_0) \right) \to \mathrm{H}_{*-1}^{l_1}\left(\coprod_\alpha B\pi_1(Y_\alpha, y_\alpha) \right) \to \\ &\mathrm{H}_{*-1}^{l_1}(B\pi_1(X, x_0)), \end{aligned} \tag{A.7}$$

along with arrows from the first sequence to the second sequence so that the diagram commutes. From [27], the latter arrows (other than the middle one) are isomorphisms. As $\mathrm{H}_*^{l_1}(X,Y)$ is isomorphic to $\mathrm{H}_*^{l_1}(Y \to X)$, the proposition follows from the five lemma. □

Remark A.8. It would follow from the relative mapping theorem stated in [15, Section 4.1], along with [27, Theorem 1.1], that the isomorphism in Proposition A.4 is an isometry. However, there doesn't seem to be a proof of the relative mapping theorem in the literature [14, Remark 4.9].

A.2. **Simplicial volume and curvature.**

Conjecture A.9. [16, Section 3.A] *For each $n \in \mathbb{Z}^+$, there is some $c_n > 0$ so that if M is a compact connected oriented n-dimensional Riemannian manifold with $R \geq -\sigma^2$ then $\parallel M \parallel \leq c_n \sigma^n \operatorname{vol}(M)$.*

One can think of Conjecture A.9 two ways. If we fix $\sigma = 1$ then $\parallel M \parallel$ would give an obstruction to volume-collapsing with a lower scalar curvature bound. If we fix $\operatorname{vol}(M) = 1$ then $\parallel M \parallel$ would give an obstruction for a manifold to have a Riemannian metric with normalized volume and almost nonnegative scalar curvature. We will think of it in the latter way.

The Ricci analog of Conjecture A.9 is known [15]. In addition there is a gap theorem; there is some $\epsilon = \epsilon(n) > 0$ so that if $\operatorname{vol}(M) = 1$ and $\operatorname{Ric} \geq -\epsilon$ then $\parallel M \parallel = 0$. On the other hand, Conjecture A.9 is not even known when $\sigma = 0$.

An analog of Conjecture A.9 for macroscopic scalar curvature, and a gap result, are known [5].

We look at whether Conjecture A.9 can be verified under some additional geometric bounds. We first prove a gap result when there is an upper diameter bound and a lower bound on the curvature operator.

Proposition A.10. *Given $n \in \mathbb{Z}^+$ and $D, \Lambda < \infty$, there is an $\epsilon = \epsilon(n, D, \Lambda) > 0$ with the following property. Let M be a compact connected spin manifold of dimension n whose fundamental group satisfies the Strong Novikov Conjecture for the maximal group C^*-algebra. Suppose that g is a Riemannian metric on M so that (M, g) has*

- *Diameter bounded above by D,*
- *Curvature operator Rm bounded below by $-\Lambda$, and*
- *Scalar curvature R bounded below by $-\epsilon$.*

Then the simplicial volume $\parallel M \parallel$ vanishes.

Proof. Suppose that the theorem is not true. Then there is a sequence of Riemannian n-manifolds $\{(M_i, g_i)\}_{i=1}^{\infty}$ so that

- $\operatorname{diam}(M_i, g_i) \leq D$,
- $\operatorname{Rm}(M_i, g_i) \geq -\Lambda$, and
- $R(M_i, g_i) \geq -\frac{1}{i}$, but
- $\parallel M_i \parallel > 0$.

The lower bound on Rm implies a lower bound on the Ricci curvature. From [15, Section 0.5], the nonvanishing of the simplicial volume then implies that there is some $v_0 > 0$ so that $\operatorname{vol}(M_i, g_i) \geq v_0$ for all i.

From [4], there are some $\tau > 0$ and $C < \infty$ so that there are Ricci flow solutions $g_i(t)$, $t \in [0, \tau]$, with

- $g_i(0) = g_i$,
- $\operatorname{Rm}(M_i, g_i(t)) \geq -C\Lambda$, and
- $|\operatorname{Rm}(M_i, g_i(t))| \leq \frac{C}{t}$.

Put $g_i' = g_i(\tau)$. The estimate $|\operatorname{Rm}(M_i, g_i(t))| \leq \frac{C}{t}$, along with Shi's derivative estimates, implies that we have uniform bounds on the k^{th}-covariant derivatives of $\operatorname{Rm}(M_i, g_i')$. It also implies a uniform upper bound $\operatorname{diam}(M_i, g_i') \leq D'$, by distance distortion estimates for the Ricci flow [25, Section 27]. By the monotonicity of scalar curvature under Ricci flow, we have $R(M_i, g_i') \geq -\frac{1}{i}$.

Applying [15, Section 0.5] again, there is some $v_0' > 0$ so that $\operatorname{vol}(M_i, g_i') \geq v_0'$ for all i. After passing to a subsequence, there is a smooth limit $\lim_{i \to \infty}(M_i, g_i') = (M_\infty, g_\infty)$, where M_∞ is diffeomorphic to each M_i, and $R(M_\infty, g_\infty) \geq 0$.

Either (M_∞, g_∞) is Ricci flat or, after running the Ricci flow, we can assume that it has positive scalar curvature. If (M_∞, g_∞) is Ricci flat then its fundamental group is virtually abelian and so $\parallel M_\infty \parallel = 0$, which is a contradiction. Hence (M_∞, g_∞) has positive scalar curvature.

If n is even then the Lichnerowicz formula implies that the index of the Dirac operator on (M_∞, g_∞) vanishes in $\mathrm{K}_0(C^*_{max}\pi_1(M_\infty))$. Let $\nu : M_\infty \to B\pi_1(M_\infty)$ be the classifying map for the universal cover of M_∞; it is defined up to homotopy. Let $[M_\infty]_K \in \mathrm{K}_n(M_\infty)$ denote the fundamental class of M_∞ in K-homology. From the Strong Novikov Conjecture, the image $\nu_*[M_\infty]_K$ vanishes in $\mathrm{K}_n(B\pi_1(M_\infty); \mathbb{Q})$. Applying the Chern character gives that $\nu_*(*\widehat{A}(TM_\infty))$ vanishes in $\mathrm{H}_*(B\pi_1(M_\infty); \mathbb{Q})$. As $*1$ equals the fundamental class $[M_\infty] \in \mathrm{H}_n(M_\infty; \mathbb{Q})$, we obtain that $\nu_*[M_\infty]$ vanishes in $\mathrm{H}_n(B\pi_1(M_\infty); \mathbb{Q})$. If n is odd then applying the preceding argument to $M_\infty \times S^1$ shows that $\nu_*[M_\infty]$ again vanishes in $\mathrm{H}_n(B\pi_1(M_\infty); \mathbb{Q})$.

Hence $\nu_*[M_\infty]$ vanishes in the l_1-homology group $\mathrm{H}_n^{l_1}(B\pi_1(M_\infty))$. From [27], the fundamental class $[M_\infty]$ now vanishes in $\mathrm{H}_n^{l_1}(M_\infty)$. Thus the simplicial volume $\parallel M_\infty \parallel$ vanishes, which is a contradiction, since M_∞ is diffeomorphic to M_i. □

Corollary A.11. *Let M be a compact connected spin manifold whose fundamental group satisfies the Strong Novikov Conjecture for the maximal group C^*-algebra. If M admits a Riemannian metric with nonnegative scalar curvature then $\parallel M \parallel = 0$.*

Remark A.12. Instead of using the Strong Novikov Conjecture, the conclusion of Proposition A.10 also holds under the assumption that $\mathrm{K}^*_{af}(B\pi_1(M)) \otimes \mathbb{Q} = \mathrm{K}^*(B\pi_1(M)) \otimes \mathbb{Q}$. This result also follows from Proposition A.10 when the Strong Novikov Conjecture for $C^*_{\max}$ is satisfied, using [21].

Remark A.13. One cannot remove the lower bound on the curvature operator in Proposition A.10, as every closed manifold of dimension greater than two admits Riemannian metrics for which $(\max |R|) \cdot \operatorname{diam}^2$ is arbitrarily small [29].

We now look at what would be involved in proving a gap theorem saying that for each $\Lambda < \infty$, there is some $\epsilon = \epsilon(n, \Lambda) > 0$ so that if a closed connected n-dimensional Riemannian manifold (M, g) has $\operatorname{vol}(g) = 1$, $R(g) \geq -\epsilon$ and $|\operatorname{Rm}|(g) \leq \Lambda$ then $\parallel M \parallel = 0$. We follow a contradiction argument along the lines of the proof of Proposition A.10. The case when the diameter stays bounded follows from Proposition A.10, so we look at the case when the diameter goes to infinity. The next proposition says that there is a thick-thin decomposition in which the thick part is modelled by a region in a complete noncompact finite volume Riemannian manifold with *positive* scalar curvature.

Proposition A.14. *Given $n \in \mathbb{Z}^+$, $v > 0$ and $\Lambda < \infty$ let $\{(M_i, g_i)\}$ be a sequence of a closed connected n-dimensional Riemannian manifolds with*

- $\mathrm{vol}(M_i, g_i) = 1$,
- $R(g_i) \geq -\frac{1}{i}$,
- $|\mathrm{Riem}|(g_i) \leq \Lambda^2$ *and*
- $\sup_i \mathrm{diam}(M_i, g_i) = \infty$.

Put $M_i^{\geq v} = \{m \in M_i \, : \, \mathrm{vol}(B(m,1)) \geq v\}$. Then after passing to a subsequence, for some N there are

- *A sequence of metrics $\{g_i'\}$ that are 1.1-biLipschitz to $\{g_i\}$,*
- *Points $\{m_{i,j}\}_{j=1}^N$ in $M_i^{\geq v}$ and*
- *A collection $\{(Z_j, \widehat{g}_j, p_j)\}_{j=1}^N$ of pointed connected complete finite volume noncompact Riemannian n-manifolds with bounded curvature and positive scalar curvature,*

so that for each $1 \leq j \leq N$, we have $\lim_{i\to\infty}(M_i, g_i', m_{i,j}) = (Z_j, p_j)$ in the pointed smooth topology, by approximants $\phi_{i,j} : (U_{i,j} \subset M_i) \to (V_{i,j} \subset Z_j)$ with $M_i^{\geq v} \subset \bigcup_{j=1}^N U_{i,j}$.

Proof. After passing to a subsequence, we can assume that $\lim_{i\to} \mathrm{diam}(M_i, g_i) = \infty$. From the double sided curvature bound, we can run the Ricci flow for a uniform time $\Delta > 0$ on each M_i, to obtain a metric g_i'. Hence we can assume that there are uniform bounds $|\nabla^I \mathrm{Riem}| \leq C_{|I|}$ on the (M_i, g_i')'s. Also, $R(g_i') \geq -\frac{1}{i}$. By taking Δ small enough, we can ensure that each g_i' is 1.1-biLipschitz to g_i.

For each i, choose a maximal collection B_i of disjoint unit g_i-balls with center in $M_i^{\geq v}$. The number of such balls is bounded above by $\lceil v^{-1} \rceil$.

If B_i is empty for all large i then we put $N = 0$ and stop. If not then after passing to a subsequence, for each i we pick some $m_{i,1} \in M_i$ so that $B(m_{i,1}, 1)$ is an element of B_i. After passing to a subsequence, we can assume that there is a smooth pointed limit $\lim_{i\to\infty}(M_i, g_i', m_{i,1}) = (M_{\infty,1}, g_{\infty,1}, m_{\infty,1})$; the fact that $\mathrm{vol}(B(m_{i,1}, 1) \geq v$ implies a lower bound on the g_i'-volume of the unit g_i'-ball around $m_{i,1}$.

We now look whether after passing to a subsequence, for each i we can find an element $B(m_{i,2}, 1)$ of B_i so that $\{d(m_{i,2}, m_{i,1})\}_{i=1}^\infty$ goes to infinity. If not, we stop and put $N = 1$. If so, we choose such new balls. After passing to a subsequence, we can assume that there is a smooth pointed limit $\lim_{i\to\infty}(M_i, g_i, m_{i,2}) = (M_{\infty,2}, g_{\infty,2}, m_{\infty,2})$.

We now look whether after passing to a subsequence, for each i we can find an element $B(m_{i,3}, 1)$ of B_i so that $\{d(m_{i,3}, \{m_{i,1}, m_{i,2}\})\}_{i=1}^\infty$ goes to infinity. If not, we stop and put $N = 2$. If so, we choose such new balls. After passing to a subsequence, we can assume that there is a smooth pointed limit $\lim_{i\to\infty}(M_i, g_i, m_{i,3}) = (M_{\infty,3}, g_{\infty,3}, m_{\infty,3})$.

We repeat the process, which must terminate in at most $\lceil v^{-1} \rceil$ iterations. In the end, we obtain a collection $\{(Z_j, \widehat{g}_j, p_j)\}_{j=1}^N$ of connected pointed complete noncompact Riemannian manifolds with

- $|\mathrm{Riem}(\widehat{g}_j)| \leq C_0$ and

- $R(\widehat{g}_j) \geq 0$.

As $(Z_j, \widehat{g}_j)$ is the result of applying a Ricci flow to a $W^{2,p}_{loc}$-regular Riemannian manifold, the strong maximum principle implies that it is Ricci-flat or has positive scalar curvature. It has volume at most one. Since a noncompact complete finite volume Riemannian manifold cannot be Ricci-flat [40], it follows that Z_j has positive scalar curvature.

The conclusion of the proposition now holds; note that $M_i^{\geq v}$ is contained in the union of the balls of radius 2 with the same centers as the elements of B_i. □

Proposition A.14 implies that for large i, we can write $M_i = M_i^{thick} \cup M_i^{thin}$ where M_i^{thick} and M_i^{thin} are n-dimensional submanifolds-with-boundary such that

- $\partial M_i^{thick} \subset \mathrm{int}(M_i^{thin})$ and $\partial M_i^{thin} \subset \mathrm{int}(M_i^{thick})$,
- M_i^{thick} is 1.1-biLipschitz equivalent to a union of regions in a finite collection of pointed connected complete finite volume noncompact Riemannian n-manifolds with bounded curvature and positive scalar curvature, and
- Points in M_i^{thin} are the centers of volume-collapsed unit balls.

Based on Corollary 4.13, we make the following hypothesis.

Hypothesis A.15. *If Z is a pointed connected complete finite volume oriented noncompact Riemannian n-manifold with bounded curvature and positive scalar curvature then there is an exhaustion $K_1 \subset K_2 \subset \dots$ of Z by compact submanifolds with the following property. Given $k \geq 1$, let $\{Y_{k,\alpha}\}$ be the connected components of $K_{k+1} - \mathrm{int}(K_k)$. Then the pushforward $\nu_*[K_{k+1}, K_{k+1} - \mathrm{int}(K_k)]$ vanishes in $\mathrm{H}_{\dim(Z)}\left(\coprod_\alpha B\pi_1(Y_{k,\alpha}) \to B\pi_1(K_{k+1}); \mathbb{Q}\right)$, where basepoints are treated as in Section 3.2.*

Assuming Hypothesis A.15, Proposition A.14 implies that we can write $M_i = M_i^{thick} \cup M_i^{thin}$ where the fundamental class $[M_i^{thick}, M_i^{thick} \cap M_i^{thin}]$ vanishes in $\mathrm{H}_n^{l_1}(M_i^{thick}, M_i^{thick} \cap M_i^{thin})$ and M_i^{thin} is locally volume-collapsed. From the lower curvature bound, there is a cover of M_i^{thin} by weakly amenable open sets with multiplicity at most n. From [23, 28], the fundamental class $[M_i^{thin}, M_i^{thick} \cap M_i^{thin}]$ vanishes in $\mathrm{H}_n^{l_1}(M_i^{thin}, M_i^{thick} \cap M_i^{thin})$. We would now like to say that $\parallel M_i \parallel = 0$. This amounts to a gluing property for simplicial volume. There are gluing results for the simplicial volume but they do not seem to apply directly.

To explain the nature of the issue, given a fine triangulation of M_i, let c be the sum of the (oriented) n-simplices and let c_{thick} be the sum of those in $\mathrm{int}(M_i^{thick})$. If the triangulation is sufficiently fine then ∂c_{thick} is a chain in $M_i^{thick} \cap M_i^{thin}$ and $(c_{thick}, -\partial c_{thick})$ represents the fundamental class in $\mathrm{H}_n^{l_1}(M_i^{thick}, M_i^{thick} \cap M_i^{thin})$. Its vanishing means that there is an l_1-chain $(d, e) \in C_{n+1}^{l_1}(M_i^{thick}, M_i^{thick} \cap M_i^{thin})$ so that $\partial d + e = c_{thick}$ and $\partial e = \partial c_{thick}$. Then $c - \partial d = c - c_{thick} + e$ is l_1-homologous to c and has support in M_i^{thin}. There are vanishing results for usual n-dimensional homology classes of M_{thin} when mapped to l_1-homology [23, 28] but they do not seem to apply to $c - \partial d$.

To phrase the issue differently, we might have made a different choice (d', e'), which would change $c - c_{thick} + e$ to $c - c_{thick} + e' = c - c_{thick} + e + (e' - e)$. As $\partial(e' - e) = 0$, we can say that $e' - e$ represents an element of $\mathrm{H}_n^{l_1}(M_i^{thick} \cap M_i^{thin})$ which lies in the image of the boundary map $\mathrm{H}^{l_1}_{n+1}(M_i^{thick}, M_i^{thick} \cap M_i^{thin}) \to \mathrm{H}^{l_1}_n(M_i^{thick} \cap M_i^{thin})$. It would seem necessary to be able to show that this element vanishes or, at least, has vanishing norm. We know a bit more about the class of $e' - e$ in $\mathrm{H}^{l_1}_n(M_i^{thick} \cap M_i^{thin})$, namely that under the isomorphism $\mathrm{H}^{l_1}_n(M_i^{thick} \cap M_i^{thin}) \cong \mathrm{H}^{l_1}_n(B\pi_1(M_i^{thick} \cap M_i^{thin}))$ it arises from an ordinary homology class in $\mathrm{H}_n(B\pi_1(M_i^{thick} \cap M_i^{thin}); \mathbb{Q})$.

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