Two readings on axioms in mathematics

1. From *Believing the Axioms* by Penelope Maddy.
Originally published in the Journal of Symbolic Logic (1988).

Believing the Axioms

Ask a beginning philosophy of mathematics student why we believe the theorems of mathematics and you are likely to hear, "because we have proofs!" The more sophisticated might add that those proofs are based on true axioms, and that our rules of inference preserve truth. The next question, naturally, is why we believe the axioms, and here the response will usually be that they are "obvious", or "self-evident", that to deny them is "to contradict oneself" or "to commit a crime against the intellect". Again, the more sophisticated might prefer to say that the axioms are "laws of logic" or "implicit definitions" or "conceptual truths"or some such thing.

Unfortunately, heartwarming answers along these lines are no longer tenable (if they ever were). On the one hand, assumptions once thought to be self-evident have turned out to be debatable, like the law of the excluded middle,[1] or outright false, like the idea that every property determines a set.[2] Conversely, the axiomatization of set theory has led to the consideration of axiom candidates that no one finds obvious, not even their staunchest supporters. In such cases, we find the methodology has more in common with the natural scientist's hypotheses formation and testing than the caricature of the mathematician writing down a few obvious truths and proceeding to draw logical consequences.

The central problem in the philosophy of natural science is when and why the sorts of facts scientists cite as evidence really are evidence. The same is true in the case of mathematics. Historically, philosophers have given considerable attention to the question of when and why various forms of logical inference are truth-preserving. The companion question of when and why the assumption of various axioms is justified has received less attention, perhaps because versions of the "self-evidence" view live on, and perhaps because of a complacent if-thenism.[3] For whatever reasons, there has been little attention to the understanding and classification of the sorts of facts mathematical scientists cite, let alone to the philosophical question of when and why those facts constitute evidence.

The question of how the unproven can be justified is especially pressing in current set theory, where the search is on for new axioms to determine the size of the continuum.[4] This pressing problem is also the deepest that contemporary mathematics presents to the contemporary philosopher of mathematics. Not only would progress towards understanding the process of mathematical hypothesis formation and confirmation contribute to our philosophical understanding of the nature of mathematics, it might even be of help and solace to those mathematicians actively engaged in the axiom search.

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[1] The law of the excluded middle says that for any proposition "P", either "P" or "not P" is true. For instance, it is always the case that either "Barack Obama is the current president of the United States" or "Barack Obama is not the current president of the United states" is true. Interestingly, contemporary logicians can give you some pretty good (but pretty complicated) reasons to not believe the law of the excluded middle!

[2] You know this one: it's Russell's paradox and the property "x is not an element of itself".

[3] "if-thenism" in this context denotes the position that you can start with whatever suppositions you want (the *if*), what warrants attention is how you draw conclusions from it (the *then*).

[4] This problem is to determine how to compare the size of the set of all real numbers with the size of other sets. We would all agree that the set $\{1, 2, 3\}$ is smaller than the set $\{1, 2, 3, 4\}$ and also smaller than the set $\{2, 5, 37, 12\}$ even though not a subset of it, but the question becomes much more delicate when there are infinitely many elements in a set. The issue is still unresolved, and today it looks like there will never be a definitive decision. Rather, mathematicians have explored two different options, leading to two different axioms about sets.

2. From *Does mathematics need new axioms?* by Solomon Feferman.
Originally published in the American Mathematical Monthly (1999).

Does mathematics need new axioms?

> My own view is that the question is an essentially philosophical one: Of course mathematics needs new axioms – we know that from Gödel's incompleteness theorems[5] – but then the questions must be: Which ones? and Why those?[6]

The Oxford English Dictionary defines 'axiom' as used in Logic and Mathematics by: "A self-evident proposition requiring no formal demonstration to prove its truth, but received and assented to as soon as mentioned." I think it's fair to say that something like this definition is the first thing we have in mind when we speak of axioms for mathematics: I'll call this the ideal sense of the word. It's surprising how far the meaning of axiom has become stretched from the ideal sense in practice, both by mathematicians and logicians. Some have even taken it to mean an arbitrary assumption and so refuse to take seriously what status axioms are to hold.

When the working mathematician speaks of axioms, he or she usually means those for some particular part of mathematics such as groups, rings, vector spaces, topological spaces, Hilbert spaces, and so on[7]. These kinds of axioms have nothing to do with self-evident propositions, nor are they arbitrary starting points. They are simply definitions of kinds of structures which have been recognized to recur in various mathematical situations. I take it that the value of these kinds of structural axioms for the organization of mathematical work is now indisputable. Moreover, we seem to keep coming up with axioms of this sort, and I thick the case can be made that they come up due to a continuing need to package and communicate our knowledge in digestible ways.

[5]Gödel's theorems are why the mathematicians working with Russell eventually had to abandon their project to do all mathematics in terms of set theory. Gödel showed that, no matter what system of axioms for mathematics you start with, there will always some true fact of mathematics (a fact not leading to a contradiction of your axioms) which you cannot prove using the axioms alone.

[6]This text is from another paper of Feferman, discussing the same question

[7]You don't need to know what these things are (Hilbert spaces, etc) in order to get the gist of Solomon's argument. You could say "geometry" instead.

Questions:

1. In a few sentences, briefly summarize Maddy's and Feffermans position in these excerpts.

2. Both Maddy and Fefferman contrast a given idea of the role of (or definition of) axioms in mathematics with the way that mathematicians actually think of or use axioms. How do both writers portray the different definitions of axioms? How do these compare to the definitions of axioms that we discussed in class?

3. What problems might arise for mathematicians if axioms are as easily created as Fefferman thinks?

4. The excerpt from Maddy is the introduction to a much longer paper, which has become a classic in the field of philosophy of mathematics. Speculate on the content of the rest of her paper.