CSP problems 3

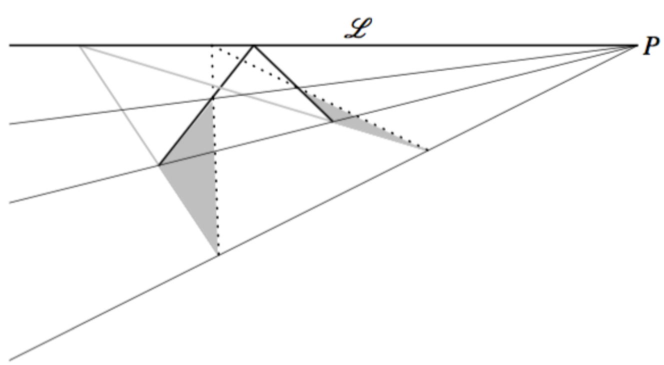
Reading: Chapters 2 and 3 (as much as you feel like) of Richter-Gerbert

More projective planes. Abstractly, we define a projective plane to be a collection of objects called the"points", and a collection of *sets* of points, called the "lines," satisfying

i) Any two points are contained in a unique line.
ii) Any two lines contain a unique point.
iii) There exist four points, no three of which are in the same line.

Problems:

1. Verify that $\mathbb{R}\mathrm{P}^2$ and the Fano plane (and more generally the $\mathbb{Z}/p\mathbb{Z}$ projective plane) satisfy the properties of a projective plane. (note that in the $\mathbb{Z}/p\mathbb{Z}$ projective plane case, already we are using "lines" to mean something a little weird, since they are finite collections of points.)
2. Show that any projective plane (satisfying the definitions above) has at least seven points. (So the Fano plane is the smallest one!) Note: remarkably, we do not completely understand what numbers of points projective planes can have. The number is always of the form n^2+n+1, but which values of n are possible is not completely figured out. See discussion in Richter-Gerbert at the end of ch. 3.
3. The *Moulton plane* is defined as follows:
   points: Usual points in $\mathbb{R}^2$, plus one extra point (think "at infinity") for each family of non-intersecting lines (where lines are defined below) *lines:* These include the usual horizontal and vertical lines, and all lines of negative slope. Each other "line" is a bent line, consisting of a half-line of slope $k > 0$ above the x-axis attached to a half-line of slope $k/2$ below the x-axis. Each line contains the points contained in it in $\mathbb{R}^2$ plus one point at infinity.
   (a) Draw some examples of lines. What are the (maximal) families of non-intersecting lines (i.e. what are the points at infinity).
   (b) Verify that the Moulton plane is a projective plane according to the definition above.
   (c) Does perspective drawing make sense in the Moulton plane? What does it look like? Draw some cubes in 1 or 2 point perspective...
   (d) Can you describe projective transformations?
4. Can you think of any variations on the Moulton plane?
5. Here is a special case of *Desargue's theorem*: Suppose you have two triangles in $\mathbb{R}^2$ (shown in grey) so that there vertices are on the same projection lines from a point (P). Show that the corresponding sides of the triangles meet at three points which all lie on the same line.



6. (a problem I meant to give you earlier!) Recall that, on problem sheet one, we showed that any continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that sends lines to lines and preserves incidence is a linear

map. Show the projective analog of this:
Suppose $f : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2$ *is a continuous function sending lines to lines and preserving incidence. Show that* f *is a projective transformation.*