CSP problems 2: the Fano plane and friends

- (fun, but worth reading for the application) the reading called "ellenberg.pdf" from Jordan Ellenberg's *How not to be wrong*
- `https://en.wikipedia.org/wiki/Fano_plane`

Finite fields. A *field* is a number system with operations of addition, subtraction, multiplication, and division. For example, usual (real) numbers. Another example is the *rational* numbers (all the fractions, i.e. numbers of the form $\frac{p}{q}$ where p, q are integers). Another good example is integers mod p where p is a prime number. These are the numbers $0, 1, 2, ..., p-1$, with addition and multiplication defined by
$a + b =$ remainder of $a + b$ when divided by p
$a \cdot b =$ remainder of $a \cdot b$ when divided by p
and subtraction $a - b =$ whatever you add to b to get a
$a/b =$ whatever you multiply by b to get a. Note that, like with ordinary numbers, you are not allowed to divide by zero.

1. As a warm up, write down addition and multiplication tables mod 5. What is $3/2$ mod 5?
2. Addition and multiplication mod p make sense if p is not prime. What goes wrong with division?
3. Linear algebra makes sense using integers mod p (or using any field) – to do matrix multiplication, you just need to be able to add and multiply numbers. What is the determinant of $\begin{pmatrix} 2 & 3//1 & 4 \end{pmatrix}$ mod 5?
4. Instead of $\mathbb{R}^2$ or $\mathbb{R}^3$, you can replace the real numbers $\mathbb{R}$ with any field. (Integers mod p is written $\mathbb{Z}/p\mathbb{Z}$.). You can do vector addition in $(\mathbb{Z}/p\mathbb{Z})^2$ or $(\mathbb{Z}/p\mathbb{Z})^3$, and talk about equations of lines and planes. What points are on the line $(1, 0, 0) + t(2, 1, 3)$ in $(\mathbb{Z}/5\mathbb{Z})^3$?

Finite projective geometries.

1. The $\mathbb{Z}/p\mathbb{Z}$ projective plane is the set of lines through $(0, 0, 0)$ in $(\mathbb{Z}/p\mathbb{Z})^3$. How many points are in the $\mathbb{Z}/2\mathbb{Z}$ projective plane? What about $\mathbb{Z}/3\mathbb{Z}$? The general case $\mathbb{Z}/p\mathbb{Z}$?
2. The *Fano plane* is the $\mathbb{Z}/2\mathbb{Z}$ projective plane. (Of which you have seen a picture). A *projective transformation* of the Fano plane is any permutation of the points that sends lines to lines. Can you describe the transformations in a systematic manner? (e.g. what are all the ones that send a particular line to itself?)
3. Can you find a way to visualize projective transformations of the Fano plane?
4. Can you draw a picture representing the $\mathbb{Z}/3\mathbb{Z}$ projective plane (hopefully representing some of its symmetries) and describe some– or all) of its transformations?
5. (challenge) How many lines are there in the $\mathbb{Z}/p\mathbb{Z}$ projective plane?
6. Read the chapter from Ellenberg's book, and summarize the application of finite projective planes to lottery problems. (note: I think this is an enlightening view towards how a lot of mathematicians think about geometry in a non-visual manner).

Optional extra things to think about: Some of the properties that we need for adding and multiplying to make sense (or to behave the way ordinary numbers work) are:
a) $0 + x = x$, and $0 \cdot x = 0$ for any x
b) $1 \cdot x = x$ for any $x \neq 0$.
c) Every number x has something (but only one thing!) you can multiply it by to get 1 (this is another way to define division – that number is $1/x$).
d) Similarly, every number has something (but only one thing!) you add to it to get 0.
Why are all of these satisfied for addition mod p?