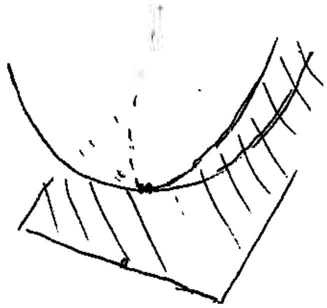
# Hyperbolic Space

## 1. Hyperboloid model.

- Lorenzian $\langle x, y \rangle$
- Hyperboloid model $\{x \mid \langle x, x \rangle = -1\} \subset \mathbb{R}^{n+1}$
- Riemannian metric defined by restriction of $\langle \, , \, \rangle$ to tangent space, tangent space to $x \in \mathbb{H}^n$ is $\{y \mid \langle x, y \rangle = 0\}$.

Isometries: – need to preserve hyperboloid and preserve metric.

Prop: Equal to $O^+(n,1)$. $O^+(n,1) \subset \mathrm{Isom}$ is clear, if $f$ is an isom, compose with element of $O^+(n,1)$ so preserves $(\vec{0}, 1)$. Then has the form $\begin{pmatrix} \Box & 0 \\ 0 & 1 \end{pmatrix} \in O^+(n,1)$



Examples:

Reflection in a hyperplane: take $W$ linear subspace of $\mathbb{R}^{n+1}$

$$R_W = \begin{cases} I & \text{on } W \\ -I & \text{on } W^\perp \end{cases}$$ for Lorenzian product.

Prop: Set of reflections generates $O^+(n,1)$; in fact reflections in $(n-1)$ dim'l hyperplanes

Proof: Isometries fixing point $p \cong O(n)$.
Show Reflections act transitively on points.

- 1-subspaces $\Leftrightarrow$ complete geodesics; & geodesic from $p$ in direction $v$ is $\gamma(t) = \cosh(t)\, p + \sinh(t)\, v$.
- Unique geodesic between any 2 points.

## 2. Poincaré disc.

- Euclidean metric scaled by $\left( \frac{2}{1 - \|x\|^2} \right)^2$ on open unit disc
- Conformal: angles are same as Euclidean angles.

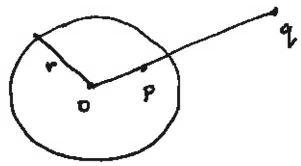
$\mathbb{D}^n \longleftrightarrow$ Hyperboloid bijection via projection thru $(\vec{0}, -1)$


(Not proved) This is an isometry.

- K-dim'l hyperplanes are intersection of $\mathbb{D}^n$ with k-sphere or k-plane in $\mathbb{R}^n$ that is orthogonal to $\partial \mathbb{D}^n$.

- Picture: angle sum $< \pi$.

Sphere inversion :




$p \mapsto q$

where $\|op\|\,\|oq\| = r^2$

This is a conformal$^{\pm}$ map.
(preserves angle but not oriented angle)

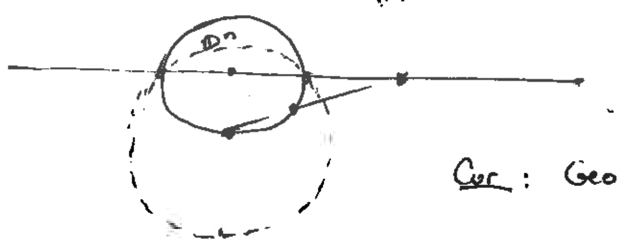
$\boxed{3.}$ Half space model :

$\mathbb{H}^n = \{(x_1, \dots x_n) \mid x_n > 0\}$ with metric $\frac{1}{x_n^2} \cdot$ Euclidean $g$. (Also conformal)

Map $\mathbb{D}^n \to \mathbb{H}^n$ via circle inversion,

This is an isometry.

$\mathbb{H}^n$



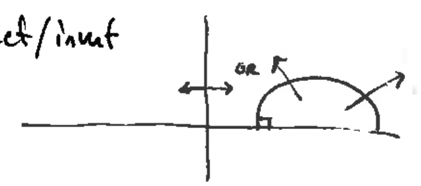

Cor : Geodesics in $\mathbb{H}^n$ are 

Isometries in $\mathbb{H}^n$ :

- Translate horizontally 
- $x \mapsto \lambda x$ for $\lambda > 0$

Proof : metric scaling $\frac{1}{x_n^2}$ vs. length scaling of a vector by this linear map cancel out exactly.

- Reflect/invert




Mauro — Mostow notes.