Context/history

References: R. Spatzier, surveys
"An invitation to rigidity theory" 2004
"Harmonic analysis in rigidity theory" 1995
(intro sections)

[Q] What is a rigidity theorem?

[A.] Something that says 'unexpected extra structure'
e.g. (3 main forms)

a) A deformation / perturbation of a system/object is equivalent to the original system/object

b) An object with a weak structure is forced to have a strong structure

c) A weak/partial isomorphism between two objects implies a strong/full isomorph.

Starting point for us...: 1960's

- Selberg 1960: suppose $n \geq 3$, $\Gamma < SL(n, \mathbb{R})$ discrete ($\exists$ nbhd $\mathcal{U}$ of id in $SL(n,\mathbb{R})$ s.t. $\Gamma \cap \mathcal{U} = \{e\}$)
$SL(n,\mathbb{R})/\Gamma$ compact. "Γ is cocompact lattice"

(a) If Γ_t a continuous deformation of Γ (entrywise on matrices)
then $\exists\, g_t \in SL(n, \mathbb{R})$ continuous family s.t. $\Gamma_t = g_t \Gamma g_t^{-1}$.

Calabi '61: similar for $\Gamma < SO(n,1)$ $n \geq 3$
$\quad = \mathrm{Isom}(\mathbb{H}^n)$

Weil '62: Generalize to all irreducible lattices in semi-simple Lie groups w/o compact factors
cocompact. not locally $SL_2\mathbb{R}$

Possible topic: Proof of Calabi–Weil. (assuming Mostow to cover rank 1 case)

[Don't write]: ['70 Garland, Raghunathan: same for irred lattices of co-finite volume
some $G \neq SL_2\mathbb{C}$ either.]

[Rank 1 vs. higher rank : negative vs nonpositive curvature]
dimension of maximal tori, diagonal group

- 1961 N. Berger (very diff't techniques – differential geometric, positive curvature)
  "Pinched curvature thm": M^{2k} Riemannian manifold, sectional curv. between 1 and 4, then
  (b) M homeomorphic to S^{2k} or
  M isometric to $\mathbb{C}P^k$, $\mathbb{H}P^{k/2}$ 'quaternionic proj. space'
  $\mathrm{Ca}\,P^{k/8}$ Cayley/octonion . . .
  (all the Rank 1 compact projective spaces)

[~~Furstenberg (probabilistic arguments) '67 → lattice in $SL_n\mathbb{R}$ can't be one in $SO(n,1)$~~] DON'T SAY

↑ say something . . .

Mostow '68:

M_1 M_2 closed hyperbolic dim $n \geq 3$, $\pi_1(M_1) \xrightarrow[\varphi]{\cong} \pi_1(M_2)$. Then $\exists!$ isometry

(c) $f: M_1 \to M_2$ s.t. φ is map induced by f.

EQUIV: Γ_1 Γ_2 cocompact lattices in $SO(n,1)$ $n \geq 1$ $\Gamma_1 \xrightarrow{\cong} \Gamma_2$ then $\cong$ is inner automorphism of $SO(n,1)$.

Set of $n \times n$ matrices preserving bilinear form
$\langle \vec{x}, \vec{y} \rangle = x_0 y_0 - \sum_{i=1}^{n} x_i y_i$

Cor: If M_1, M_2 are h.e. then they are isometric.

Rk: There are many examples of mflds of same dimension where h.e. doesn't even imply homeomorphic, e.g. Lens spaces.

'73 Prasad generalizes to co-finite volume

'73 Mostow generalizes to irred. cocompact lattices in connected SS Lie gp w/o cpt factor

'79 Totall new proof of Mostow by Gromov. (cocompact case)

Influence : — large scale geometry of hyperbolic space ⇝ Geometric group theory
(Gromov "Hyperbolic groups '87

— "Boundary Maps", probabilistic methods, tools for Homogeneous dynamics (Groups acting on locally symmetric spaces)

— Rigidity of groups / group actions :

Margulis superrigidity '79 : Γ irred. in SS etc. Lie gp G
(special case)
$\Gamma \xrightarrow{\varphi} H$ some Lie alg group, Zariski dense

Then φ extends ! to continuous homomorphism $G \longrightarrow H$

All proofs of original Mostow have common set up:

Take $M_1 \xrightarrow{f} M_2$ h.e. (M_i is a $K(\pi,1)$)

lift to $\widetilde{M}_1 \xrightarrow{\tilde{f}} \widetilde{M}_2$, show: extends to continuous map on compactification of $\mathbb{H}^n$ by S^{n-1},
$\widetilde{M}_1 = \mathbb{H}^n$, $\widetilde{M}_2 = \mathbb{H}^n$

study regularity of this map (proofs diverge!)

Show: same as induced by an isometry,