

# RATIONAL STRUCTURES AND $(\varphi, \Gamma)$ -MODULES

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Let  $K$  be a finite extension of the field  $\mathbb{Q}_p$  of  $p$ -adic numbers. For smooth proper algebraic varieties over  $K$ , Grothendieck conjectured the existence of a *mysterious functor* relating étale cohomology with  $\mathbb{Z}_p$ -coefficients and algebraic de Rham cohomology. A precise formulation of this conjecture was given by Fontaine: for  $X$  a smooth proper  $K$ -variety, there should be a  *$p$ -adic comparison isomorphism*

$$H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbf{B}_{\text{dR}} \cong H_{\text{dR}}^i(X, K) \otimes_K \mathbf{B}_{\text{dR}}$$

in which  $\mathbf{B}_{\text{dR}}$  is a certain topological  $K$ -algebra equipped with an action of the Galois group  $G_K$  and a descending filtration, and the isomorphism is compatible with Galois actions and filtrations (using the Hodge filtration on de Rham cohomology) on both sides. One obtains the mysterious functor by forming the left side of the comparison isomorphism and then taking Galois invariants. The conjecture of Fontaine, and a more precise conjecture of Fontaine and Janssen that makes it possible to recover étale cohomology with  $\mathbb{Q}_p$ -coefficients from de Rham cohomology, have been established using various techniques by various authors, including Faltings, Tsuji, Nizioł, Scholze, Beilinson, and Bhatt. See the introduction to [4] for references.

The étale cohomology group appearing in the  $p$ -adic comparison isomorphism may be viewed as an object in the category of finite  $\mathbb{Z}_p$ -modules with continuous  $G_K$ -action. Fontaine also observed that this category can be described equivalently in terms of  $(\varphi, \Gamma)$ -modules, which are finite modules over a certain topological ring with continuous semilinear actions by a certain *commutative* monoid. One can refine this description by making suitable choices of the base ring, as shown by Cherbonnier-Colmez, Berger, et al.; this makes it possible to recover de Rham cohomology of a smooth proper variety over  $K$  directly from the  $(\varphi, \Gamma)$ -module associated to  $p$ -adic étale cohomology. See [3] for references.

By formally combining the two previous paragraphs, one obtains a functor from smooth proper  $K$ -varieties to  $(\varphi, \Gamma)$ -modules from which one may recover the  $p$ -adic comparison isomorphism. There are good reasons to view this functor as an independent cohomology theory in its own right; for instance, Colmez [5] has discovered a mechanism for explicating the local Langlands correspondence for the group  $\text{GL}_2(\mathbb{Q}_p)$  using  $(\varphi, \Gamma)$ -modules.

The purpose of this paper is to provide a different reason: the construction of  $(\varphi, \Gamma)$ -modules can be refined so as to accommodate rational structures on de Rham cohomology. Suppose now that  $K$  is a number field equipped with a choice of a  $p$ -adic place  $v$ , so that the completion  $K_v$  is again a finite extension of  $\mathbb{Q}_p$ . For a smooth proper variety  $X$  over  $K$ , the algebraic de Rham cohomology of  $X_{K_v}$  can be viewed naturally as the base extension of the algebraic de Rham cohomology of  $X$  itself. Consequently, to the  $(\varphi, \Gamma)$ -module associated to  $X_{K_v}$  one may wish to add the extra datum of a descent of the de Rham cohomology from

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*Date:* August 22, 2013.

Thanks to Chris Davis and Peter Scholze for feedback. Supported by NSF grant DMS-1101343, DARPA grant HR0011-09-1-0048, UC San Diego (Warschawski Professorship).

$K_v$  to  $K$  (compatible with the Hodge filtration). This latter datum amounts to a descent on the Sen representation of the  $(\varphi, \Gamma)$ -module; what we show is that the category of  $(\varphi, \Gamma)$ -modules equipped with such descent data can be described in terms of  $(\varphi, \Gamma)$ -modules (or more precisely  $(\varphi^{-1}, \Gamma)$ -modules) over a somewhat smaller base ring.

We also describe a refinement of this construction when  $K = \mathbb{Q}$  and  $X$  has good reduction at  $v = p$ . In that case, the de Rham comparison isomorphism descends to a smaller period ring  $\mathbf{B}_{\text{crys}}$  admitting a Frobenius automorphism  $\varphi$ , there is a canonical isomorphism between  $H_{\text{dR}}^i(X, \mathbb{Q}_p)$  and the crystalline cohomology  $H_{\text{crys}}^i(X, \mathbb{Q}_p)$ , and the resulting three-way isomorphism

$$H_{\text{et}}^i(X_{\overline{\mathbb{Q}_p}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbf{B}_{\text{crys}} \cong H_{\text{dR}}^i(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\text{crys}} \cong H_{\text{crys}}^i(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\text{crys}}$$

is compatible with the Frobenius action on  $H_{\text{crys}}^i(X, \mathbb{Q}_p)$  in addition to the other extra structures. One can capture all three structures using a modified version of  $(\varphi, \Gamma)$ -modules called *Wach modules*; in case  $X$  is defined over  $\mathbb{Q}$ , one can again modify the base ring for the Wach module construction so as to keep track of the  $\mathbb{Q}$ -rational structure on de Rham cohomology.

Our preferred (if still somewhat fanciful) interpretation of the aforementioned constructions is as a first step towards a global theory of  $(\varphi, \Gamma)$ -modules which would provide a coherent interpolation of  $p$ -adic Hodge theory across all primes  $p$ , including the infinite prime. For a complementary approach, see [6].

## 1. $(\varphi, \Gamma)$ -MODULES

We begin by recalling the usual theory of  $(\varphi, \Gamma)$ -modules in the form needed for this paper. Our notations follow [11], to which we refer for further background and references.

**Remark 1.1.** For ease of exposition, we present everything in this paper over the base field  $\mathbb{Q}_p$ , rather than allowing finite extensions. For the comparison between de Rham and étale cohomology, one may easily accommodate finite extensions using Weil restriction on the algebraic geometry side and Shapiro's lemma on the Galois representation side. For the comparison with crystalline cohomology in the good reduction case, extension fields present some complications which we choose not to address (but see Remark 6.10).

**Definition 1.2.** Equip the Laurent series field  $\mathbb{F}_p((\overline{\pi}))$  with the  $\overline{\pi}$ -adic absolute value  $|\bullet|$  with the normalization  $|\overline{\pi}| = p^{-p/(p-1)}$ . Extend this absolute value multiplicatively to the perfect closure  $\mathbb{F}_p((\overline{\pi}))^{\text{perf}}$ , then complete to get the field  $\tilde{\mathbf{E}}$ . Let  $\tilde{\mathbf{E}}^+$  be the valuation subring of  $\tilde{\mathbf{E}}$ .

Let  $\tilde{\mathbf{A}}$  (resp.  $\tilde{\mathbf{A}}^+$ ) be the ring of  $p$ -typical Witt vectors with coefficients in  $\tilde{\mathbf{E}}$  (resp.  $\tilde{\mathbf{E}}^+$ ). The elements of  $\tilde{\mathbf{A}}$  (resp.  $\tilde{\mathbf{A}}^+$ ) can be described as infinite sums  $\sum_{n=0}^{\infty} [\overline{x}_n] p^n$  with  $\overline{x}_n \in \tilde{\mathbf{E}}$  (resp.  $\tilde{\mathbf{E}}^+$ ), where the brackets denote the Teichmüller lifting. For  $r > 0$ , we define the function  $|\bullet|_r : \tilde{\mathbf{A}} \rightarrow [0, +\infty]$  by the formula

$$\left| \sum_{n=0}^{\infty} [\overline{x}_n] p^n \right|_r = \sup_n \{ p^{-n} |\overline{x}_n|^r \}.$$

Let  $\tilde{\mathbf{A}}^{\dagger, r-}$  be the subset of  $\tilde{\mathbf{A}}$  on which  $|\bullet|_r$  takes finite values. Let  $\tilde{\mathbf{A}}^{\dagger, r}$  be the subset of  $\tilde{\mathbf{A}}^{\dagger, r-}$  consisting of those elements  $\sum_{n=0}^{\infty} [\overline{x}_n] p^n$  for which  $\lim_{n \rightarrow \infty} p^{-n} |\overline{x}_n|^r = 0$ . These are

both subrings of  $\tilde{\mathbf{A}}$  on which  $|\bullet|_r$  is a complete multiplicative nonarchimedean norm [11, Lemma 1.7.2]. Define  $\tilde{\mathbf{A}}^\dagger = \bigcup_{r>0} \tilde{\mathbf{A}}^{\dagger,r} = \bigcup_{r>0} \tilde{\mathbf{A}}^{\dagger,r-}$ .

Let  $\mathbf{A}$  be the  $p$ -adic completion of  $\mathbb{Z}_p((\pi))$ , viewed as a subring of  $\tilde{\mathbf{A}}$  by identifying  $1 + \pi$  with  $[1 + \bar{\pi}]$ . For  $*$   $\in \{\emptyset; \dagger, r; \dagger, r-; \dagger; +\}$ , put  $\mathbf{A}^* = \mathbf{A} \cap \tilde{\mathbf{A}}^*$ ; note that  $\mathbf{A}^+ = \mathbb{Z}_p[[\pi]]$ .

We next introduce some operators on the rings we have just constructed.

**Definition 1.3.** Let  $\varphi : \tilde{\mathbf{E}} \rightarrow \tilde{\mathbf{E}}$  denote the  $p$ -power Frobenius map. Put  $\Gamma = \mathbb{Z}_p^\times$  and consider the unique continuous action of  $\Gamma$  on  $\tilde{\mathbf{E}}$  for which  $\gamma(1 + \bar{\pi}) = (1 + \bar{\pi})^\gamma$  for all  $\gamma \in \Gamma$ . By the functoriality of the Witt vector construction, we obtain actions of  $\varphi$  and  $\Gamma$  on  $\tilde{\mathbf{A}}$  commuting with Teichmüller lifting. These actions descend to subrings as follows.

- The action of  $\Gamma$  preserves all of  $\tilde{\mathbf{A}}, \tilde{\mathbf{A}}^{\dagger,r}, \tilde{\mathbf{A}}^{\dagger,r-}, \tilde{\mathbf{A}}^\dagger, \tilde{\mathbf{A}}^+, \mathbf{A}, \mathbf{A}^{\dagger,r}, \mathbf{A}^{\dagger,r-}, \mathbf{A}^\dagger, \mathbf{A}^+$ .
- The action of  $\varphi$  carries  $\tilde{\mathbf{A}}^{\dagger,r}$  and  $\tilde{\mathbf{A}}^{\dagger,r-}$  into  $\tilde{\mathbf{A}}^{\dagger,r/p}$  and  $\tilde{\mathbf{A}}^{\dagger,(r/p)-}$ , respectively. Consequently,  $\varphi$  preserves  $\tilde{\mathbf{A}}, \tilde{\mathbf{A}}^\dagger, \tilde{\mathbf{A}}^+, \mathbf{A}, \mathbf{A}^\dagger, \mathbf{A}^+$ .
- The map  $\varphi$  is bijective on  $\tilde{\mathbf{A}}$ . Its inverse preserves  $\tilde{\mathbf{A}}, \tilde{\mathbf{A}}^{\dagger,r}, \tilde{\mathbf{A}}^{\dagger,r-}, \tilde{\mathbf{A}}^\dagger, \tilde{\mathbf{A}}^+$ .

**Definition 1.4.** For any ring  $R$ , we will say that an  $R$ -module  $M$  is *finite  $p$ -free* if the  $p$ -torsion submodule  $T$  of  $M$  is finitely generated and  $M/T$  is finite free.

**Remark 1.5.** For many of the rings  $R$  we consider here, finite  $p$ -free modules are the same as finitely generated modules. However, for some of the rings we consider, we lack enough ring-theoretic analysis to say for sure whether this equality occurs; for instance, it may be that finitely generated modules are only *finite  $p$ -projective*.

**Definition 1.6.** Take  $R$  to be any subring of  $\tilde{\mathbf{A}}$  stable under  $\varphi$  and  $\Gamma$ . A  $(\varphi, \Gamma)$ -*module* over  $R$  is a finite  $p$ -free  $R$ -module  $M$  equipped with commuting semilinear actions of  $\varphi$  and  $\Gamma$  such that the action of  $\varphi$  induces an isomorphism  $\varphi^*M \cong M$ .

**Remark 1.7.** What we are calling a  $(\varphi, \Gamma)$ -module here is more precisely an *étale*  $(\varphi, \Gamma)$ -module: under some circumstances one may prefer to assume only that the map  $\varphi^*M \rightarrow M$  is an isogeny (meaning that its kernel and cokernel are killed by some power of  $p$ ). Since only étale  $(\varphi, \Gamma)$ -modules arise in the context of comparison isomorphisms, we will not introduce the more general definition here.

**Remark 1.8.** In the literature, it is common to assume that the underlying  $R$ -module of a  $(\varphi, \Gamma)$ -module is free. We allow  $p$ -torsion here (as in Fontaine's original definition) in order to account for torsion in cohomology; the technical impact on the development of the theory is minimal.

The theory of  $(\varphi, \Gamma)$ -modules is then summarized by the following statements.

**Theorem 1.9.** (a) *The categories of  $(\varphi, \Gamma)$ -modules over  $\mathbf{A}, \mathbf{A}^\dagger, \tilde{\mathbf{A}}, \tilde{\mathbf{A}}^\dagger$  are all equivalent via the natural base extension functors.*

(b) *The categories in (a) are all equivalent to the category of finite  $\mathbb{Z}_p$ -modules with continuous  $G_{\mathbb{Q}_p}$ -actions.*

*Proof.* Note that the rings  $\mathbf{A}, \mathbf{A}^\dagger, \tilde{\mathbf{A}}, \tilde{\mathbf{A}}^\dagger$  are all discrete valuation rings with maximal ideals generated by  $p$ . The equivalences between  $(\varphi, \Gamma)$ -modules over  $\mathbf{A}$ ,  $(\varphi, \Gamma)$ -modules over  $\tilde{\mathbf{A}}$ , and finite  $\mathbb{Z}_p$ -modules with continuous  $G_{\mathbb{Q}_p}$ -actions can all be checked at the level of torsion modules, for which see [8] or [11, Theorem 2.3.5]. The equivalences between  $(\varphi, \Gamma)$ -modules

over  $\mathbf{A}^\dagger, \tilde{\mathbf{A}}^\dagger, \tilde{\mathbf{A}}$  are automatic for torsion modules because  $\mathbf{A}^\dagger$  and  $\tilde{\mathbf{A}}^\dagger$  are  $p$ -adically dense in  $\mathbf{A}$  and  $\tilde{\mathbf{A}}$ , respectively. It thus suffices to check these equivalences in the torsion-free case, for which see [11, Theorem 2.4.5, Theorem 2.6.2], respectively.  $\square$

**Remark 1.10.** Although we will not use it explicitly in this paper, it is worth pointing out the construction of the functor from  $(\varphi, \Gamma)$ -modules over  $\tilde{\mathbf{A}}$  to Galois representations whose existence is implied by Theorem 1.9. From that theorem, we obtain a functor turning each finite étale  $\mathbb{Q}_p$ -algebra  $L$  into a finite étale  $\tilde{\mathbf{E}}$ -algebra  $\tilde{\mathbf{E}}_L$  carrying actions of  $\varphi$  and  $\Gamma$ . Then given a  $(\varphi, \Gamma)$ -module  $M$  over  $\tilde{\mathbf{E}}$ , the associated  $G_{\mathbb{Q}_p}$ -representation is

$$D_{\text{et}}(M) = \varprojlim_{n \rightarrow \infty} \bigcup_L (M \otimes W(\tilde{\mathbf{E}}_L)/(p^n))^{\varphi, \Gamma},$$

where  $L$  runs through the finite Galois extensions of  $\mathbb{Q}_p$  within a fixed algebraic closure of  $\mathbb{Q}_p$  whose Galois group has been identified with  $G_{\mathbb{Q}_p}$ .

**Example 1.11.** Let  $\chi : G_{\mathbb{Q}_p} \rightarrow \Gamma$  denote the cyclotomic character; that is, for any  $p$ -th power root of unity  $\zeta$  and any  $\gamma \in \Gamma$ ,

$$\gamma(\zeta) = \zeta^{\chi(\gamma)}.$$

Then the  $(\varphi, \Gamma)$ -module corresponding to  $\chi$  over any of  $\mathbf{A}, \mathbf{A}^\dagger, \tilde{\mathbf{A}}, \tilde{\mathbf{A}}^\dagger$  is free on one generator  $\mathbf{v}$  satisfying

$$\varphi(\mathbf{v}) = \mathbf{v}, \quad \gamma(\mathbf{v}) = \gamma^{-1} \cdot \mathbf{v} \quad (\gamma \in \Gamma).$$

We will also need the following variant of Theorem 1.9.

**Definition 1.12.** Take  $R$  to be any subring of  $\tilde{\mathbf{A}}$  stable under  $\varphi^{-1}$  and  $\Gamma$ . We define a  $(\varphi^{-1}, \Gamma)$ -module over  $R$  by replacing  $\varphi$  with  $\varphi^{-1}$  in the definition of a  $(\varphi, \Gamma)$ -module. In case  $R$  is stable under both  $\varphi$  and  $\varphi^{-1}$ , it is easily seen that the categories of  $(\varphi, \Gamma)$ -modules and  $(\varphi^{-1}, \Gamma)$ -modules over  $R$  are equivalent.

**Theorem 1.13.** For any  $r > 0$ , the categories of  $(\varphi^{-1}, \Gamma)$ -modules over

$$\tilde{\mathbf{A}}, \tilde{\mathbf{A}}^\dagger, \bigcup_{n=0}^{\infty} \varphi^{-n}(\mathbf{A}), \bigcup_{n=0}^{\infty} \varphi^{-n}(\mathbf{A}^\dagger), \tilde{\mathbf{A}}^{\dagger, r}, \tilde{\mathbf{A}}^{\dagger, r^{-}}, \bigcup_{n=0}^{\infty} \varphi^{-n}(\mathbf{A}^{\dagger, rp^{-n}}), \bigcup_{n=0}^{\infty} \varphi^{-n}(\mathbf{A}^{\dagger, rp^{-n}-})$$

are equivalent via the natural base extension functors. Consequently (by Theorem 1.9), they are all equivalent to the category of finite  $\mathbb{Z}_p$ -modules with continuous  $G_{\mathbb{Q}_p}$ -actions.

*Proof.* Since  $\tilde{\mathbf{A}}, \bigcup_{n=0}^{\infty} \varphi^{-n}(\mathbf{A}), \tilde{\mathbf{A}}^\dagger, \bigcup_{n=0}^{\infty} \varphi^{-n}(\mathbf{A}^\dagger)$  are stable under both  $\varphi$  and  $\varphi^{-1}$ , the equivalence for these four rings follows from Theorem 1.9. The other rings may be each be added to the chain of equivalences by the following process, which we demonstrate explicitly for the ring  $\tilde{\mathbf{A}}^{\dagger, r}$  and leave implicit in the other cases.

We check that base extension along  $\tilde{\mathbf{A}}^{\dagger, r} \rightarrow \tilde{\mathbf{A}}$  is fully faithful and essentially surjective. For full faithfulness, it suffices (by taking internal Homs) to check that for any  $(\varphi^{-1}, \Gamma)$ -module  $M$  over  $\tilde{\mathbf{A}}^{\dagger, r}$ , any element  $\mathbf{v}$  of  $M \otimes_{\tilde{\mathbf{A}}^{\dagger, r}} \tilde{\mathbf{A}}$  fixed by  $\varphi^{-1}$  and  $\Gamma$  belongs to  $M$  itself. By the previous paragraph,  $\mathbf{v}$  must belong to  $M \otimes_{\tilde{\mathbf{A}}^{\dagger, r}} \tilde{\mathbf{A}}^\dagger$  and hence to  $M \otimes_{\tilde{\mathbf{A}}^{\dagger, r}} \tilde{\mathbf{A}}^{\dagger, s}$  for some  $s > 0$ . Using the isomorphism  $(\varphi^{-1})^* M \cong M$ , we deduce that  $\mathbf{v}$  also belongs to  $M \otimes_{\tilde{\mathbf{A}}^{\dagger, r}} \tilde{\mathbf{A}}^{\dagger, s'}$  for  $s' = \min\{r, ps\}$ ; after finitely many iterations, we deduce that  $\mathbf{v} \in M$ .

For essential surjectivity, it suffices to check that any  $(\varphi^{-1}, \Gamma)$ -module over  $\tilde{\mathbf{A}}$  can be descended to  $\tilde{\mathbf{A}}^{\dagger, r}$ . Again by the first paragraph, we obtain a descent to  $\tilde{\mathbf{A}}^{\dagger, s}$  for some  $s > 0$ . We may then pull back along  $\varphi^{-1}$  to replace  $s$  with  $\min\{r, ps\}$  as needed.  $\square$

**Remark 1.14.** In addition to the aforementioned constructions, there is also an important variation in which one inverts  $p$  and then takes a certain Fréchet completion. For example, starting with  $\mathbf{A}^{\dagger}$ , this yields the *Robba ring* of germs of analytic functions on open annuli of outer radius 1 over  $\mathbb{Q}_p$ . (There is also an analogue starting from  $\tilde{\mathbf{A}}^{\dagger}$  for which  $\varphi$  is bijective.) By the work of Berger, many important constructions in  $p$ -adic Hodge theory can be factored through the construction of  $(\varphi, \Gamma)$ -modules over the Robba ring, including the crystalline Frobenius and monodromy operators on de Rham cohomology. We will not consider this topic further here, but see [3] for more discussion.

## 2. THE THETA MAP

We next recall the definition of the *theta map* in  $p$ -adic Hodge theory.

**Definition 2.1.** Let  $\zeta_0, \zeta_1, \dots$  be a sequence of elements of some algebraic closure of  $\mathbb{Q}$  such that  $\zeta_n$  is a primitive  $p^n$ -th root of unity and  $\zeta_{n+1}^p = \zeta_n$ . Let  $\mathbb{Q}(\mu_{p^\infty})$  be the algebraic extension of  $\mathbb{Q}$  generated by the  $\zeta_n$ . Let  $K$  be the completion of  $\mathbb{Q}(\mu_{p^\infty})$  for the unique multiplicative extension of the  $p$ -adic absolute value. Using the cyclotomic character, we may identify  $\Gamma$  with  $\text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \cong \text{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ , and thus obtain an action of  $\Gamma$  also on  $K$ .

**Definition 2.2.** For  $r > 0$  and  $x \in \tilde{\mathbf{A}}^{\dagger, r}$  nonzero, the *Newton polygon* of  $x$  is obtained by writing  $x = \sum_{n=0}^{\infty} [\bar{x}_n] p^n$ , taking the lower convex hull of the set  $\{(-\log_p |\bar{x}_n|, n) : n = 0, 1, \dots\}$ , and omitting any slopes outside of the range  $[-r, 0)$ . The Newton polygon has the expected multiplicativity property: for  $x, y$  nonzero, the Newton polygon of  $xy$  is obtained by joining the Newton polygons of  $x$  and  $y$  and then sorting the segments by slope to obtain a convex polygonal line [10, Lemma 2.1.7]. Using Newton polygons, one can show that  $\tilde{\mathbf{A}}^{\dagger, r}$  is a euclidean domain and hence a principal ideal domain [10, Proposition 2.6.5]. The same arguments apply to  $\mathbf{A}^{\dagger, r}$  for  $r \leq 1$ .

**Remark 2.3.** The ring  $\mathbf{A}^+$  is not a principal ideal domain; rather, it is a two-dimensional local ring. However, the ring  $\mathbf{A}^+[p^{-1}]$  is a principal ideal domain.

**Definition 2.4.** Put

$$\pi_n = \frac{[1 + \bar{\pi}^{p^{-n}}] - 1}{[1 + \bar{\pi}^{p^{-n-1}}] - 1} \quad (n \in \mathbb{Z}).$$

Then there is a continuous  $\Gamma$ -equivariant homomorphism  $\theta : \tilde{\mathbf{A}}^{\dagger, 1} \rightarrow K$  such that  $\theta(\pi_n) = \zeta_{p^n} - 1$  for  $n = 0, 1, \dots$  and  $\ker(\theta) = (\pi_0)$ ; see [11, Lemma 1.3.3] for the construction. For  $n \in \mathbb{Z}$ , let  $\theta_n : \tilde{\mathbf{A}}^{\dagger, p^n} \rightarrow K$  denote the composition  $\theta \circ \varphi^n$ .

**Lemma 2.5.** *For any nonzero  $x \in \tilde{\mathbf{A}}^{\dagger, 1}$ , there exists  $n_0$  such that  $\theta_{-n}(x) \neq 0$  for all  $n \geq n_0$ . In particular, the product map  $\theta_0 \times \theta_{-1} \times \dots$  on  $\tilde{\mathbf{A}}^{\dagger, 1}$  is injective.*

*Proof.* If the claim were false for some  $x$ , then  $x$  would be divisible by  $\varphi^n(\pi_0)$  for infinitely many nonnegative integers  $n$ . However, this would force the Newton polygon of  $x$  to have infinitely many distinct slopes, which is impossible.  $\square$

**Definition 2.6.** Let  $\mathbf{B}_{\mathrm{dR}, \mathbb{Q}_p}^+$  be the  $\ker(\theta)$ -adic completion of  $\tilde{\mathbf{A}}^{\dagger, 1}$ ; this is a complete discrete valuation ring with residue field  $K$  on which  $\Gamma$  acts, but  $\varphi$  does not. One convenient generator for the maximal ideal of  $\mathbf{B}_{\mathrm{dR}, \mathbb{Q}_p}^+$  is  $t = \log(1 + \pi)$ , which has the property that  $\gamma(t) = \gamma \cdot t$  for all  $\gamma \in \Gamma$ . Let  $\mathbf{B}_{\mathrm{dR}, \mathbb{Q}_p}$  be the fraction field of  $\mathbf{B}_{\mathrm{dR}, \mathbb{Q}_p}^+$ , equipped with the descending  $t$ -adic filtration:  $\mathrm{Fil}^i \mathbf{B}_{\mathrm{dR}, \mathbb{Q}_p} = t^i \mathbf{B}_{\mathrm{dR}, \mathbb{Q}_p}^+$ .

**Remark 2.7.** The ring  $\mathbf{B}_{\mathrm{dR}, \mathbb{Q}_p}$  is the  $\ker(G_{\mathbb{Q}_p} \rightarrow \Gamma)$ -invariant part of Fontaine's de Rham period ring  $\mathbf{B}_{\mathrm{dR}}$ , which appears in the standard formulation of the  $p$ -adic comparison isomorphism (as in the introduction).

**Remark 2.8.** The Cohen structure theorem implies the existence of an isomorphism  $\mathbf{B}_{\mathrm{dR}, \mathbb{Q}_p}^+ \cong K[[t]]$ , but there is no natural choice of such an isomorphism.

### 3. DESCENT OF $(\varphi^{-1}, \Gamma)$ -MODULES

We next use the theta map to define some smaller base rings for  $(\varphi, \Gamma)$ -modules.

**Lemma 3.1.** *For any nonnegative integer  $m$ , the subring of  $\tilde{\mathbf{A}}^{\dagger, 1}$  consisting of those elements  $x$  for which  $\theta_{-n}(x) \in \mathbb{Q}_p(\zeta_{p^{n+m}})$  for  $n = 0, 1, \dots$  is equal to  $\varphi^{-m}(\mathbf{A}^{\dagger, p^{-m}})$ .*

*Proof.* Put  $T = \{x \in \tilde{\mathbf{A}}^{\dagger, 1} : \theta_{-n}(x) \in \mathbb{Q}_p(\zeta_{p^n}) \ (n = 0, 1, \dots)\}$ . On one hand, it is clear that  $\varphi^{-m}(\mathbf{A}^{\dagger, p^{-m}}) \subseteq T$ . On the other hand, suppose that  $x \in \tilde{\mathbf{A}}^{\dagger, 1} \setminus \varphi^{-m}(\mathbf{A}^{\dagger, p^{-m}})$ ; there then exists a largest nonnegative integer  $j$  such that  $x = y + p^j z$  for some  $y \in \varphi^{-m}(\mathbf{A}^{\dagger, p^{-m}})$ ,  $z \in \tilde{\mathbf{A}}^{\dagger, 1}$ . Write  $z = \sum_{i=0}^{\infty} [\bar{z}_i] p^i$  with  $\bar{z}_i \in \tilde{\mathbf{E}}$ ; by the maximality of  $j$ , we must have  $\bar{z}_0 \notin \mathbb{F}_p((\bar{\pi}^{p^{-m}}))$ . In fact, we can even choose  $z$  so that  $|\bar{z}_0|$  is not an integral power of  $|\bar{\pi}^{p^{-m}}|$ . Then for  $n$  sufficiently large,  $|\theta_{-n}(z)| = |\bar{z}_0|^{p^{-n}}$  is not the norm of an element of  $\mathbb{Q}_p(\zeta_{p^{n+m}})$ , so  $z \notin T$  and hence  $x \notin T$ .  $\square$

**Definition 3.2.** Let  $\tilde{S}_p$  denote the subring of  $\tilde{\mathbf{A}}^{\dagger, 1}$  consisting of those elements  $x$  for which  $\theta_{-n}(x) \in \mathbb{Q}_p(\mu_{p^\infty})$  for all  $n \geq 0$ . Also put  $S_p = \bigcup_{m=0}^{\infty} \varphi^{-m}(\mathbf{A}^{\dagger, p^{-m}})$ ; by Lemma 3.1,  $S_p$  can be characterized directly as the subring of  $\tilde{\mathbf{A}}^{\dagger, 1}$  consisting of those elements  $x$  for which there exists  $m = m(x)$  such that  $\theta_{-n}(x) \in \mathbb{Q}_p(\zeta_{p^{m+n}})$  for all  $n$ . In particular,  $S_p$  is a subring of  $\tilde{S}_p$ . The rings  $S_p$  and  $\tilde{S}_p$  are stable under  $\varphi^{-1}$  and  $\Gamma$ , so we may consider  $(\varphi^{-1}, \Gamma)$ -modules over them.

**Remark 3.3.** Let  $R$  be one of  $S_p, \tilde{S}_p$ . For  $x, y \in R$  with  $\theta_{-n}(y) \neq 0$  for all  $n \geq 0$ ,  $x$  is divisible by  $y$  in  $R$  if and only if  $x$  is divisible by  $y$  in  $\tilde{\mathbf{A}}^{\dagger, 1}$ .

**Lemma 3.4.** *For any positive integer  $j$ , the maps*

$$S_p/p^j S_p \rightarrow \tilde{S}_p/p^j \tilde{S}_p \rightarrow \tilde{\mathbf{A}}^{\dagger, 1}/p^j \tilde{\mathbf{A}}^{\dagger, 1}$$

*are isomorphisms.*

*Proof.* Injectivity of each map follows from Remark 3.3 applied with  $y = p^j$ . Surjectivity reduces to the case  $j = 1$ , which follows by observing that for each positive integer  $m$ , the reduction modulo  $p$  of  $\varphi^{-1}(\mathbf{A}^{\dagger, p^{-m}})$  equals  $\mathbb{F}_p((\bar{\pi}^{p^{-m}}))$ . (This is not true for  $m = 0$  because  $\pi^{-1} \notin \tilde{\mathbf{A}}^{\dagger, 1}$ .)  $\square$

**Theorem 3.5.** *The categories of  $(\varphi^{-1}, \Gamma)$ -modules over  $S_p, \tilde{S}_p, \tilde{\mathbf{A}}^{\dagger, 1}$  are equivalent via base extension.*

*Proof.* The equivalence between  $S_p$  and  $\tilde{\mathbf{A}}^{\dagger,1}$  is included in Theorem 1.13; it thus suffices to check that base extension from  $\tilde{S}_p$  to  $\tilde{\mathbf{A}}^{\dagger,1}$  is fully faithful. By Lemma 3.4, we may ignore  $p$ -torsion and consider only objects whose underlying modules are free. By forming internal Homs, it further suffices to verify that for  $M$  a  $(\varphi^{-1}, \Gamma)$ -module over  $\tilde{S}_p$  whose underlying module is free, any  $(\varphi^{-1}, \Gamma_{\mathbb{Q}})$ -stable element  $\mathbf{v}$  of  $M \otimes_{\tilde{S}_p} \tilde{\mathbf{A}}^{\dagger,1}$  belongs to  $\tilde{S}_p$ .

We first note that the image of  $\mathbf{v}$  in  $M \otimes_{\theta} K$  is forced to belong to  $M \otimes_{\theta} \mathbb{Q}_p(\mu_{p^\infty})$  by Sen's decompletion theorem [13]. We then note that  $\mathbf{v} \otimes_{\theta_{-n}} \mathbb{Q}_p(\mu_{p^\infty}) = \varphi^{-n}(\mathbf{v}) \otimes_{\theta} \mathbb{Q}_p(\mu_{p^\infty}) = \mathbf{v} \otimes_{\theta} \mathbb{Q}_p(\mu_{p^\infty})$ , so if we write  $\mathbf{v}$  in terms of a basis of  $M$ , the coefficients all belong to  $\tilde{S}_p$ . This proves the claim.  $\square$

#### 4. FURTHER DESCENT VIA RATIONAL STRUCTURES

We next introduce an even smaller base ring which can be used to keep track of rational structures.

**Definition 4.1.** Put  $B^+ = \mathbb{Q}(\mu_{p^\infty})[[t]]$ ,  $B_p^+ = \mathbb{Q}_p(\mu_{p^\infty})[[t]]$ ,  $B = B^+[t^{-1}]$ , and  $B_p = B_p^+[t^{-1}]$ . Equip  $B$  and  $B_p$  with the decreasing  $t$ -adic filtrations. We define an action of  $\Gamma$  on  $B_p$  by the formula  $\gamma(\sum c_n t^n) = \sum \gamma(c_n) t^n$ . The subgroup  $\Gamma_{\mathbb{Q}} = \mathbb{Z}_p^\times \cap \mathbb{Q}$  of  $\Gamma$  also acts on  $B$ .

We may identify  $B_p^+$  with the  $\pi$ -adic completion of  $S_p$ . The inclusion  $S_p \rightarrow \tilde{\mathbf{A}}^{\dagger,1}$  then induces a  $\Gamma$ -equivariant inclusion  $B_p \rightarrow \mathbf{B}_{\text{dR}, \mathbb{Q}_p}$ . On the other hand, the obvious inclusion  $B \rightarrow B_p$  is equivariant for the action of  $\Gamma_{\mathbb{Q}}$  but not for  $\Gamma$ , which does not act on  $B$ .

**Remark 4.2.** The existence of a distinguished inclusion  $B_p \rightarrow \mathbf{B}_{\text{dR}, \mathbb{Q}_p}$  may be a bit confusing in light of Remark 2.8. The point here is that the projection  $B_p^+ \rightarrow \mathbb{Q}_p(\mu_{p^\infty})$  only admits one  $\mathbb{Q}_p$ -algebra section by Hensel's lemma, but no such uniqueness property holds for the projection  $\mathbf{B}_{\text{dR}, \mathbb{Q}_p}^+ \rightarrow K$ .

**Definition 4.3.** Let  $S'$  be the subring of  $S_p$  consisting of those elements  $x$  for which  $\theta_{-n}(x) \in \mathbb{Q}(\mu_{p^\infty})$  for all  $n$ . The ring  $S$  is stable under  $\varphi^{-1}$  and  $\Gamma$ , so we can define  $(\varphi^{-1}, \Gamma)$ -modules over  $S'$ .

Let  $S$  be the subring of  $S_p$  consisting of those elements  $x$  for which the image of  $\varphi^{-n}(x)$  in  $B_p^+$  belongs to  $B^+$  for all  $n$ ; note that  $S$  is contained in  $S'$ . The ring  $S$  is stable under  $\varphi^{-1}$  and  $\Gamma_{\mathbb{Q}}$  but not  $\Gamma$ , so we cannot define  $(\varphi^{-1}, \Gamma)$ -modules over  $S$ ; see Definition 4.8 instead.

**Remark 4.4.** In  $B_p^+$ , we have by definition  $t = \log(1 + \pi)$ , so

$$\pi = \exp(t) - 1 = \sum_{n=1}^{\infty} \frac{t^n}{n!} \in B^+.$$

Consequently, for any positive integer  $m$ ,  $[1 + \bar{\pi}]^{p^{-m}} - 1$  (viewed as an element of  $B_p^+$ ) is a root of the monic polynomial  $P(T) = T^{(p-1)p^{m-1}} + \dots + T^{p^{m-1}} + 1 - \pi$  over  $B^+$ . Since the reduction of  $P(T)$  modulo  $t$  has distinct roots (namely the primitive  $p^m$ -th roots of unity), by Hensel's lemma we have  $[1 + \bar{\pi}]^{p^m} - 1 \in B^+$ . Consequently,  $\pi_m \in S$  for  $m = 0, 1, \dots$ .

For  $\gamma \in \Gamma_{\mathbb{Q}}$ , we have  $\gamma(1 + \pi) \in S$  by the previous paragraph. By contrast, for  $\gamma \in \Gamma \setminus \Gamma_{\mathbb{Q}}$ , we have  $\gamma(1 + \pi) \in S' \setminus S$  because the image of  $\gamma(1 + \pi)$  in  $B_p^+/t^2 B_p^+$  is  $1 + \gamma t$ .

**Remark 4.5.** The analogue of Remark 3.3 holds for  $R = S$  and for  $R = S'$ : for  $x, y \in R$  with  $\theta_{-n}(y) \neq 0$  for all  $n \geq 0$ ,  $x$  is divisible by  $y$  in  $R$  if and only if  $x$  is divisible by  $y$  in  $\tilde{\mathbf{A}}^{\dagger,1}$ .

The key observation that makes our descent argument work is that the definition of  $S$  within  $\tilde{\mathbf{A}}^{\dagger,1}$  involves restrictions modulo powers of the  $\pi_n$ , and that these restrictions are in some sense independent from both the  $p$ -adic topology and the norm topology. This observation is formalized in the following lemma.

**Lemma 4.6.** *Take  $R = \varphi^{-h}(\mathbf{A}^+)$  or  $R = \varphi^{-h}(\mathbf{A}^{\dagger,p^{-h}})$  for some positive integer  $h$ . For  $n \geq 0$ , let  $R_n$  be the  $\ker(\theta_{-n})$ -adic completion of  $R$ . Let  $M$  be a finite free module over  $R$  equipped with the supremum norm (relative to  $|\bullet|_1$ ) for some basis. For  $n, k \geq 0$ , equip  $M_{n,k} = M \otimes_R (R_n / \ker(\theta_{-n})^k)$  with the quotient topology. For  $n \geq 0$ , choose a quantity  $\epsilon_n > 0$  and a subset  $V_n$  of  $M_{n,\infty} = \varprojlim_{k \rightarrow \infty} M_{n,k}$  such that for each  $k \geq 0$  and each  $\mathbf{w}$  in the image of  $V_n \rightarrow M_{n,k}$ , the image of  $V_n \rightarrow M_{n,k+1}$  is dense in the inverse image of  $\mathbf{w}$  in  $M_{n,k+1}$ . Choose also a nonnegative integer  $m$ , a quantity  $\delta > 0$ , and an element  $\mathbf{v} \in M$ . Then there exists  $\mathbf{w} \in M$  such that  $\mathbf{v} - \mathbf{w} \in p^m M$ ,  $|\mathbf{v} - \mathbf{w}| < \delta$ , and for each nonnegative integer  $n$ , the image of  $\mathbf{w}$  in  $M_{n,\infty}$  belongs to  $V_n$  and  $|\theta_{-n}(\mathbf{v} - \mathbf{w})| < \epsilon_n$ .*

*Proof.* We will construct elements  $\mathbf{w}_{-1}, \mathbf{w}_0, \mathbf{w}_1, \dots$  of  $M$  in such a way that  $\mathbf{v} - \mathbf{w}_n \in p^m M$ ,  $|\mathbf{v} - \mathbf{w}_n| < \delta$ , and for  $0 \leq n' \leq n$  the image of  $\mathbf{w}_n$  in  $M_{n',n-n'+1}$  belongs to the image of  $V_{n'} \rightarrow M_{n',n-n'+1}$  and  $|\theta_{-n'}(\mathbf{v} - \mathbf{w}_n)| < \epsilon_{n'}$ . We will also ensure that  $|\mathbf{w}_n - \mathbf{w}_{n-1}| \leq p^{-n}$  for  $n \geq 0$ ; then the  $\mathbf{w}_n$  will converge to a limit  $\mathbf{w}$  having the desired property.

To begin, put  $\mathbf{w}_{-1} = \mathbf{v}$ . To continue, suppose we are given  $\mathbf{w}_{n-1}$  for some  $n \geq 0$ . Since  $R[p^{-1}]$  is a principal ideal domain (see Definition 2.2 and Remark 2.3), we can find  $a_i, b_i \in R$  for  $i = 0, \dots, n$  and  $e \geq m$  for which

$$a_i \left( \prod_{j \leq n, j \neq i} \pi_j^{n-j+1} \right) + b_i \pi_i = p^e.$$

Using the density property of  $V_n$ , we may choose  $\mathbf{x}_i \in M$  so that

$$\mathbf{w}_n = \mathbf{w}_{n-1} + \sum_{i=0}^n a_i \left( \prod_{j=0}^{n-1} \pi_j^{n-j} \right) \left( \prod_{j \leq n, j \neq i} \pi_j \right) \mathbf{x}_i$$

has all of the desired properties: the image of  $\mathbf{w}_n$  in  $M_{n',n-n'+1}$  depends only on  $\mathbf{x}_{n'}$ , and the density property allows us to make  $\mathbf{x}_{n'}$  small enough to enforce the norm restrictions.  $\square$

**Corollary 4.7.** *For any positive integer  $m$ , the map*

$$S/p^m S \rightarrow S_p/p^m S_p$$

*is an isomorphism.*

*Proof.* Injectivity follows from Remark 4.5 applied with  $y = p^m$ . Surjectivity follows from Lemma 4.6.  $\square$

**Definition 4.8.** Equip the group  $\Gamma_{\mathbb{Q}}$  with the subspace topology from  $\Gamma$ . A  $(\varphi^{-1}, \Gamma_{\mathbb{Q}})$ -module over  $S$  is a finite  $p$ -free  $S$ -module  $M$  equipped with commuting semilinear actions of  $\varphi$  and  $\Gamma_{\mathbb{Q}}$  such that the action of  $\varphi$  induces an isomorphism  $\varphi^* M \cong M$  and the action of  $\Gamma_{\mathbb{Q}}$  on  $M \otimes_S S_p$  extends to an action of  $\Gamma$ . The second condition is equivalent to requiring the action of  $\Gamma_{\mathbb{Q}}$  on  $M$  to be continuous with respect to the supremum norm (relative to  $|\bullet|_1$ ) for some basis; this implies in turn that the action of  $\Gamma_{\mathbb{Q}}$  on  $M \otimes_S \tilde{\mathbf{A}}$  is continuous for the weak topology on  $\tilde{\mathbf{A}}$ , so the extension to  $M \otimes_S S_p$  is unique if it exists.



**Definition 4.9.** Let  $M$  be a  $(\varphi^{-1}, \Gamma)$ -module over  $R = S'$  or  $R = S_p$ . By a *rational descent datum* on  $M$ , we will mean a finite free module  $N$  over  $B^+$  equipped with a semilinear  $\Gamma_{\mathbb{Q}}$ -action and a  $\Gamma_{\mathbb{Q}}$ -equivariant isomorphism  $M \otimes_R B_p^+ \cong N \otimes_{B^+} B_p^+$ . For instance, if  $M$  arises by base extension from a  $(\varphi^{-1}, \Gamma_{\mathbb{Q}})$ -module  $M_0$  over  $S$ , then  $M_0 \otimes_S B^+$  provides a rational descent datum on  $M$ .

**Theorem 4.10.** (a) *The categories of  $(\varphi^{-1}, \Gamma)$ -modules over  $S'$  and  $S$  are equivalent via base extension.*

(a) *The base extension functor from  $(\varphi^{-1}, \Gamma_{\mathbb{Q}})$ -modules over  $S$  to  $(\varphi^{-1}, \Gamma)$ -modules over  $S_p$  equipped with rational descent data is an equivalence of categories.*

*Proof.* By Corollary 4.7, we may ignore  $p$ -torsion and consider only objects whose underlying modules are free. We only write out the proof of (b), the proof of (a) being similar.

Full faithfulness follows as in the proof of Theorem 3.5 using the rational descent datum in place of Sen's theorem. To check essential surjectivity, let  $M$  be a  $(\varphi^{-1}, \Gamma)$ -module over  $S_p$  equipped with a rational descent datum. This means that we have been given a  $\Gamma_{\mathbb{Q}}$ -equivariant isomorphism  $M \otimes_{S_p} B_p^+ \cong N \otimes_{B^+} B_p^+$  in which  $N$  is a finite free  $B^+$ -module equipped with a semilinear action of  $\Gamma_{\mathbb{Q}}$ . Let  $V_n$  be the pullback of  $N$  along  $\varphi^{-n}$ , viewed as a subspace of  $M \otimes_{\varphi^{-n}} B_p^+$ . Let  $M_0$  be the set of  $\mathbf{v} \in M$  for which  $\mathbf{v} \otimes_{\varphi^{-n}} B_p^+ \in V_n$  for  $n = 0, 1, \dots$ . Choose a basis  $\mathbf{e}_1, \dots, \mathbf{e}_d$  of  $M$ . By Lemma 4.6, we can construct elements  $\mathbf{e}'_1, \dots, \mathbf{e}'_d \in M_0$  such that the  $d \times d$  matrix  $A$  for which  $\mathbf{e}'_j = \sum_i A_{ij} \mathbf{e}_i$  has the property that  $|A - 1|_r < 1$  and  $|\theta_{-n}(A - 1)| < 1$  for all  $n \geq 0$ . On one hand, this means that  $\mathbf{e}'_1, \dots, \mathbf{e}'_d$  form a basis of  $M$ . On the other hand, for any  $n \geq 0$ ,  $\theta_{-n}(\mathbf{e}'_1), \dots, \theta_{-n}(\mathbf{e}'_d)$  form a basis of  $M \otimes_{\theta_{-n}} \mathbb{Q}_p(\mu_{p^\infty})$ , so the images of  $\mathbf{e}'_1, \dots, \mathbf{e}'_d$  in  $V_n$  must also form a basis. Consequently, for any  $\mathbf{v} \in M_0$ , the unique expression of  $\mathbf{v}$  as a  $\tilde{\mathbf{A}}^{\dagger,1}$ -linear combination of  $\mathbf{e}'_1, \dots, \mathbf{e}'_d$  is forced to have coefficients whose images under  $\varphi^{-n}$  map into  $B^+$  for all  $n \geq 0$ . These coefficients thus belong to  $S$ ; that is,  $\mathbf{e}'_1, \dots, \mathbf{e}'_d$  form a basis of  $M_0$ , so the latter is indeed a  $(\varphi^{-1}, \Gamma_{\mathbb{Q}})$ -module over  $S$  whose base extension is  $M$  with the chosen rational descent datum.  $\square$

## 5. RATIONAL STRUCTURES AND THE COMPARISON ISOMORPHISM

We next recall how de Rham cohomology relates to  $(\varphi, \Gamma)$ -modules, then indicate how to transport rational structures on de Rham cohomology.

**Definition 5.1.** For  $M$  a  $(\varphi^{-1}, \Gamma)$ -module over  $\tilde{\mathbf{A}}^{\dagger,1}$ , put  $D_{\text{dR}}(M) = (M \otimes_{\tilde{\mathbf{A}}^{\dagger,1}} \mathbf{B}_{\text{dR}, \mathbb{Q}_p})^{\Gamma}$ . This is a finite-dimensional  $\mathbb{Q}_p$ -vector space equipped with an exhausting decreasing filtration induced by the filtration on  $\mathbf{B}_{\text{dR}, \mathbb{Q}_p}$ . Using the fact that  $\mathbf{B}_{\text{dR}, \mathbb{Q}_p}^{\Gamma} = \mathbb{Q}_p$ , it is easy to check that the natural map  $D_{\text{dR}}(M) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\text{dR}, \mathbb{Q}_p} \rightarrow M \otimes_{\tilde{\mathbf{A}}^{\dagger,1}} \mathbf{B}_{\text{dR}, \mathbb{Q}_p}$  is injective; if it is also surjective, we say that  $M$  is *de Rham*.

For  $M$  a  $(\varphi^{-1}, \Gamma)$ -module over  $S_p$ , we say  $M$  is *de Rham* if  $M \otimes_{S_p} \tilde{\mathbf{A}}^{\dagger,1}$  is de Rham. In this case, a result of Berger [1, Proposition 5.9] shows that if we put  $D_{\text{dR}}(M) = (M \otimes_{S_p} B_p)^{\Gamma}$ , then the natural  $D_{\text{dR}}(M) \rightarrow D_{\text{dR}}(M \otimes_{S_p} \tilde{\mathbf{A}}^{\dagger,1})$  is an isomorphism.

In this language, we can formulate the  $p$ -adic de Rham comparison isomorphism theorem as originally conjectured by Fontaine. See the introduction of [4] for references.

**Theorem 5.2** (Comparison isomorphism). *Let  $X$  be a smooth proper  $\mathbb{Q}_p$ -scheme. Choose  $i \geq 0$  and equip  $V = H_{\text{et}}^i(X_{\overline{\mathbb{Q}_p}}, \mathbb{Z}_p)$  with the action of  $G_{\mathbb{Q}_p}$ . Equip the de Rham cohomology*

group  $H_{\text{dR}}^i(X, \mathbb{Q}_p)$  with the Hodge filtration, normalized as a descending filtration with

$$(5.2.1) \quad \text{Fil}^i H_{\text{dR}}^i(X, \mathbb{Q}_p) = H^0(X, \Omega_{X/\mathbb{Q}_p}^i).$$

Apply Theorem 1.9 and Theorem 1.13 to convert  $V$  into a  $(\varphi^{-1}, \Gamma)$ -module  $M$  over  $\tilde{\mathbf{A}}^{\dagger,1}$ . Then  $M$  is de Rham and there is a natural (in particular, functorial in  $X$ ) identification  $D_{\text{dR}}(M) \cong H_{\text{dR}}^i(X, \mathbb{Q}_p)$ , under which the filtration on  $D_{\text{dR}}(M)$  corresponds to the Hodge filtration.

To add the rational structure on de Rham cohomology to the comparison isomorphism, we must convert this rational structure into a rational descent datum on the  $(\varphi^{-1}, \Gamma)$ -module. This requires a lemma complementary to Sen's theorem.

**Lemma 5.3.** *Let  $V$  be a finite-dimensional  $\mathbb{Q}(\mu_{p^\infty})$ -vector space equipped with a semilinear action of  $\Gamma_{\mathbb{Q}}$  which is continuous for the  $p$ -adic topology. Suppose that  $V \otimes_{\mathbb{Q}(\mu_{p^\infty})} \mathbb{Q}_p(\mu_{p^\infty})$  admits a basis fixed by  $\Gamma_{\mathbb{Q}}$ . Then  $V$  also admits such a basis.*

*Proof.* Fix a basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  of  $V$  and another basis  $\mathbf{v}_1, \dots, \mathbf{v}_n$  of  $V \otimes_{\mathbb{Q}(\mu_{p^\infty})} \mathbb{Q}_p(\mu_{p^\infty})$  fixed by  $\Gamma_{\mathbb{Q}}$ . Let  $U$  be the change-of-basis matrix from the  $\mathbf{e}_i$  to the  $\mathbf{v}_j$ . Choose  $\gamma \in \Gamma_{\mathbb{Q}}$  of infinite order. Let  $m$  be a positive integer large enough so that  $U$  has entries in  $\mathbb{Q}_p(\mu_{p^m})$  and the matrix of action of  $\gamma$  on the basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  has entries in  $\mathbb{Q}(\mu_{p^m})$ . Let  $W$  be the  $\mathbb{Q}(\mu_{p^m})$ -span of  $\mathbf{e}_1, \dots, \mathbf{e}_n$ . Choose a positive integer  $h$  for which  $\gamma^h$  has trivial image in  $\text{Gal}(\mathbb{Q}(\mu_{p^m})/\mathbb{Q})$ ; then  $\gamma^h$  acts  $\mathbb{Q}(\mu_{p^m})$ -linearly on  $W$ . However,  $W \otimes_{\mathbb{Q}(\mu_{p^m})} \mathbb{Q}_p(\mu_{p^m})$  admits the basis  $\mathbf{v}_1, \dots, \mathbf{v}_n$  fixed by  $\gamma^h$ , so  $W$  must itself be fixed by  $\gamma^h$ .

Let  $H$  be the closure in  $\Gamma$  of the subgroup generated by  $\gamma^h$ ; then  $H \cap \Gamma_{\mathbb{Q}}$  also acts trivially on  $W$ . Let  $F$  be the subfield of  $\mathbb{Q}(\mu_{p^\infty})$  fixed by  $H$ ; it is a finite extension of  $\mathbb{Q}$ . We then obtain a semilinear action of  $\Gamma_{\mathbb{Q}}/(H \cap \Gamma_{\mathbb{Q}}) \cong \text{Gal}(F/\mathbb{Q})$  on  $W \otimes_{\mathbb{Q}(\mu_{p^m})} F$ . By ‘‘Theorem 90’’ (i.e., the theorem of Hilbert-Noether-Speiser), this action admits a fixed basis, which then is a fixed basis of  $V$  under the action of  $\Gamma_{\mathbb{Q}}$ .  $\square$

**Theorem 5.4.** *Let  $M$  be a de Rham  $(\varphi^{-1}, \Gamma)$ -module over  $S_p$ . Then rational descent data on  $M$  are naturally in correspondence with  $\mathbb{Q}$ -rational structures on  $D_{\text{dR}}(M)$  compatible with the Hodge filtration.*

*Proof.* In one direction, given a  $\mathbb{Q}$ -rational structure on  $D_{\text{dR}}(M)$  compatible with the Hodge filtration, we may pick out a  $B$ -submodule of  $D_{\text{dR}}(M) \otimes_{\mathbb{Q}_p} B_p \cong M \otimes_{S_p} B_p$  (where the isomorphism comes from the result of Berger cited in Definition 5.1) and hence a  $B^+$ -submodule of  $\text{Fil}^0(D_{\text{dR}}(M) \otimes_{\mathbb{Q}_p} B_p) \cong M \otimes_{S_p} B_p^+$ . This gives rise to a rational descent datum on  $M$ .

Conversely, given a rational descent datum on  $M$ , Theorem 4.10 descends  $M$  to a  $(\varphi^{-1}, \Gamma_{\mathbb{Q}})$ -module  $M_0$  over  $S$ . We then obtain a natural inclusion

$$(5.4.1) \quad (M_0 \otimes_S B)^{\Gamma_{\mathbb{Q}}} \otimes_{\mathbb{Q}} \mathbb{Q}_p \rightarrow D_{\text{dR}}(M)$$

which we claim is an isomorphism. To check this, we may write

$$D_{\text{dR}}(M) = \varinjlim_{m \rightarrow \infty} (M \otimes_{S_p} t^{-m} B_p^+)^{\Gamma_{\mathbb{Q}}} = \varinjlim_{m \rightarrow \infty} \varprojlim_{n \rightarrow \infty} (M \otimes_{S_p} t^{-m} B_p^+ / t^{-m+n} B_p^+)^{\Gamma_{\mathbb{Q}}}$$

and then apply Lemma 5.3 to verify that

$$(M_0 \otimes_S t^{-m} B^+ / t^{-m+n} B^+)^{\Gamma_{\mathbb{Q}}} \rightarrow (M \otimes_{S_p} t^{-m} B_p^+ / t^{-m+n} B_p^+)^{\Gamma_{\mathbb{Q}}}$$

is an isomorphism. Given that (5.4.1) is an isomorphism, we obtain a  $\mathbb{Q}$ -rational structure on  $D_{\text{dR}}(M)$  compatible with the Hodge filtration.  $\square$

Putting everything together, we obtain the following theorem.

**Theorem 5.5** (Comparison isomorphism with rational structure). *Let  $X$  be a smooth proper  $\mathbb{Q}$ -scheme. Then for each  $i \geq 0$ , there is a natural (in particular, functorial in  $X$ ) way to associate to  $X$  a  $(\varphi^{-1}, \Gamma_{\mathbb{Q}})$ -module  $M_i$  over  $S$  so as to obtain natural identifications*

$$(5.5.1) \quad D_{\text{et}}(M_i \otimes_S \tilde{\mathbf{A}}) \cong H_{\text{et}}^i(X_{\overline{\mathbb{Q}_p}}, \mathbb{Z}_p)$$

of  $G_{\mathbb{Q}_p}$ -modules and

$$(5.5.2) \quad (M_i \otimes_S B)^{\Gamma_{\mathbb{Q}}} \cong H_{\text{dR}}^i(X, \mathbb{Q})$$

of filtered  $\mathbb{Q}$ -vector spaces.

*Proof.* Apply Theorem 5.2 to produce a de Rham  $(\varphi^{-1}, \Gamma)$ -module over  $\tilde{\mathbf{A}}^{\dagger,1}$ . By Theorem 3.5 and Definition 5.1, this descends uniquely to a de Rham  $(\varphi^{-1}, \Gamma)$ -module over  $S_p$ . Using the rational structure on  $H_{\text{dR}}^i(X, \mathbb{Q}_p)$  provided by  $H_{\text{dR}}^i(X, \mathbb{Q})$ , we obtain a rational descent datum and hence by Theorem 4.10 a  $(\varphi^{-1}, \Gamma_{\mathbb{Q}})$ -module over  $S$  satisfying (5.5.2). We obtain (5.5.1) from Remark 1.10.  $\square$

## 6. THE GOOD REDUCTION CASE

As noted in the introduction, for a scheme with good reduction at  $p$ , the comparison isomorphism extends to a three-way comparison between étale cohomology (with its Galois action), de Rham cohomology (with the Hodge filtration), and crystalline cohomology (with its Frobenius action). One can capture all three cohomologies at once by replacing  $(\varphi, \Gamma)$ -modules with somewhat smaller objects called *Wach modules*, as described by Berger [2]. (A closely related construction is that of *Breuil-Kisin modules*, as described in [12].)

**Definition 6.1.** Put  $q = \pi_{-1}$ . A *Wach module* over  $\mathbf{A}^+$  is a finite  $p$ -free  $\mathbf{A}^+$ -module  $M^+$  equipped with a semilinear  $\Gamma$ -action fixing  $M^+/\pi M^+$  and a  $\varphi$ -semilinear  $\Gamma$ -equivariant map  $M^+ \rightarrow M^+[q^{-1}]$  which induces an isomorphism  $(\varphi^* M^+)[q^{-1}] \cong M^+[q^{-1}]$ . Note that  $M^+ \otimes_{\mathbf{A}^+} \mathbf{A}$  may be viewed as a  $(\varphi, \Gamma)$ -module over  $\mathbf{A}$  because  $q$  is a unit in  $\mathbf{A}$ , and similarly with  $\mathbf{A}$  replaced by  $\mathbf{A}^{\dagger}$ .

**Example 6.2.** Let  $M$  be the  $(\varphi, \Gamma)$ -module over  $\mathbf{A}$  corresponding to the cyclotomic character  $\chi$  as in Example 1.11. The submodule  $\mathbf{A}^+ \mathbf{v}$  of  $M$  is stable under  $\varphi$  and  $\Gamma$ , but it is not a Wach module because the action of  $\Gamma$  on  $\mathbf{A}^+ \mathbf{v} \otimes_{\mathbf{A}^+, \theta} \theta(\mathbf{A}^+)$  is nontrivial. However,  $M^+ = \pi \mathbf{A}^+ \mathbf{v}$  is a Wach module.

**Remark 6.3.** Let  $M^+$  be a Wach module over  $\mathbf{A}^+$ . Then for any sufficiently large integer  $n$ ,  $\varphi(\pi^n M^+) \subseteq \pi^n M^+$ .

The following lemma equates our definition of a Wach module over  $\mathbf{A}^+$  with that of Berger [2, Proposition II.1.1].

**Lemma 6.4.** *Let  $M^+$  be a Wach module over  $\mathbf{A}^+$  and let  $M = M^+ \otimes_{\mathbf{A}^+} \mathbf{A}$  be the associated  $(\varphi, \Gamma)$ -module over  $\mathbf{A}$ .*

- (a) *There exists a maximal  $(\varphi, \Gamma)$ -stable finitely generated  $\mathbf{A}^+$ -submodule  $M_0$  of  $M$ .*
- (b) *For any sufficiently large integer  $n$ ,  $\pi^n M^+ \subseteq M_0$  and  $\pi^n M_0 \subseteq M^+$ .*

*Proof.* Part (a) is a theorem of Fontaine [8, Proposition II.2.1.5]. Given this, the inclusion  $\pi^n M^+ \subseteq N$  for sufficiently large  $n$  follows from Remark 6.3. Given that inclusion,  $M_0/\pi^n M^+$  is a finite torsion module over  $\mathbf{A}^+$  on which multiplication by  $p$  is bijective. By Weierstrass preparation,  $M_0/\pi^n M^+$  is isomorphic as an  $\mathbf{A}^+$ -module to a finite direct sum  $\bigoplus_i \mathbf{A}^+/(f_i^{e_i})$  for some irreducible monic polynomials  $f_i \in \mathbb{Z}_p[\pi]$  and some positive integers  $e_i$ . Because  $M_0$  and  $M^+[q^{-1}]$  both carry actions of  $\varphi$ , the set  $\{f_i\} \setminus \{\pi\}$  must be stable under  $\varphi$ ; but since this set is finite, it must be empty. It follows that  $M_0/\pi^n M^+$  is killed by a power of  $\pi$ , proving (b).  $\square$

**Lemma 6.5.** *The base extension functor from Wach modules over  $\mathbf{A}^+$  to  $(\varphi, \Gamma)$ -modules over  $\mathbf{A}$  is fully faithful.*

*Proof.* Let  $M^+, N^+$  be two Wach modules over  $\mathbf{A}^+$ , put  $M = M^+ \otimes_{\mathbf{A}^+} \mathbf{A}$ ,  $N = N^+ \otimes_{\mathbf{A}^+} \mathbf{A}$ , and let  $f : M \rightarrow N$  be a morphism of  $(\varphi, \Gamma)$ -modules. Let  $M_0, N_0$  be the submodules of  $M, N$  given by Lemma 6.4(a); we then have  $f(M_0) \subseteq N_0$ . By Lemma 6.4(b), we then have  $f(M^+) \subseteq \pi^{-n} N^+$  for some positive integer  $n$ . Suppose that  $n > 0$ . If we consider the  $\Gamma$ -equivariant homomorphism  $M^+/\pi M^+ \rightarrow \pi^{-n} N^+/\pi^{-n+1} N^+$ , we notice that  $\Gamma$  fixes the source but acts on the target via a nonzero power of the cyclotomic character. It follows that this homomorphism must vanish, so we can replace  $n$  by  $n - 1$ . That is, we have  $f(M^+) \subseteq N^+$ , proving the claim.  $\square$

**Definition 6.6.** We say that a  $(\varphi, \Gamma)$ -module  $M$  over  $\mathbf{A}$  is *crystalline* if its associated  $G_{\mathbb{Q}_p}$ -representation  $D_{\text{et}}(M)$  is crystalline in the sense of Fontaine. See [3] for more discussion of this condition.

**Theorem 6.7.** *Base extension defines an equivalence of categories between Wach modules over  $\mathbf{A}^+$  and crystalline  $(\varphi, \Gamma)$ -modules over  $\mathbf{A}$ .*

*Proof.* Since base extension from Wach modules over  $\mathbf{A}^+$  to  $(\varphi, \Gamma)$ -modules over  $\mathbf{A}$  is fully faithful by Lemma 6.5, it suffices to identify the essential image of this functor. This follows from [2, Proposition II.1.1] using Lemma 6.4 to equate our definition of Wach modules with that of Berger (and twisting using Remark 6.3).  $\square$

**Theorem 6.8** (Crystalline comparison isomorphism). *Let  $X$  be a smooth proper  $\mathbb{Z}_p$ -scheme. Then for  $i \geq 0$ , there is a natural (in particular, functorial in  $X$ ) way to associate to  $X$  a Wach module  $M_i^+$  over  $\mathbf{A}^+$  so as to obtain natural identifications*

$$(6.8.1) \quad D_{\text{et}}(M_i^+ \otimes_{\mathbf{A}^+} \tilde{\mathbf{A}}) \cong H_{\text{et}}^i(X_{\overline{\mathbb{Q}_p}}, \mathbb{Z}_p)$$

of  $G_{\mathbb{Q}_p}$ -modules,

$$(6.8.2) \quad (M_i^+ \otimes_{\mathbf{A}^+} B_p^+)^{\Gamma} \cong H_{\text{dR}}^i(X_{\mathbb{Q}_p}, \mathbb{Q}_p)$$

of filtered  $\mathbb{Q}_p$ -vector spaces for the filtration induced on the left side by the filtration

$$\text{Fil}^j M_i^+ = \{\mathbf{v} \in M_i^+ : \varphi(\mathbf{v}) \in q^j M_i^+\}$$

of  $M_i^+$  and the usual filtration on  $B_p^+$ , and

$$(6.8.3) \quad (M_i^+/\pi M_i^+) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H_{\text{crys}}^i(X_{\mathbb{F}_p}, \mathbb{Q}_p)$$

of  $\varphi$ -modules over  $\mathbb{Q}_p$ .

*Proof.* The crystalline comparison theorem, in the form proved first by Faltings [7], implies that  $H_{\text{et}}^i(X_{\overline{\mathbb{Q}_p}}, \mathbb{Z}_p)$  is crystalline and that applying Fontaine's functor  $D_{\text{crys}}$  to it yields  $H_{\text{crys}}^i(X_{\mathbb{F}_p}, \mathbb{Q}_p)$  as a  $\varphi$ -module. By Theorem 6.7, the  $(\varphi, \Gamma)$ -module  $M_i$  over  $\mathbf{A}$  corresponding to  $H_{\text{et}}^i(X_{\overline{\mathbb{Q}_p}}, \mathbb{Z}_p)$  via Theorem 1.9 descends to a Wach module  $M_i^+$  over  $\mathbf{A}^+$ . This module satisfies (6.8.1) and (6.8.2) thanks to the de Rham comparison isomorphism (Theorem 5.4) and satisfies (6.8.3) thanks to [2, Théorème III.4.4].  $\square$

**Remark 6.9.** One can take the Wach module  $M_i^+$  over  $\mathbf{A}^+$  appearing in Theorem 6.8 and form the base extension  $\tilde{M}_i^+ = M_i^+ \otimes_{\mathbf{A}^+} \tilde{\mathbf{A}}^+$  to obtain what one might call a *Wach module* over  $\tilde{\mathbf{A}}^+$ . One can then formulate Theorem 6.8 directly in terms of  $\tilde{M}_i^+$ , but a bit of care is required because  $\pi$  is not irreducible in  $\tilde{\mathbf{A}}^+$ .

The identification of étale cohomology remains essentially unchanged:

$$(6.9.1) \quad D_{\text{et}}(\tilde{M}_i^+ \otimes_{\tilde{\mathbf{A}}^+} \tilde{\mathbf{A}}) \cong H_{\text{et}}^i(X_{\overline{\mathbb{Q}_p}}, \mathbb{Z}_p).$$

The identification of de Rham cohomology is also similar:

$$(6.9.2) \quad (\tilde{M}_i^+ \otimes_{\tilde{\mathbf{A}}^+} \mathbf{B}_{\text{dR}, \mathbb{Q}_p}^+)^{\Gamma} \cong H_{\text{dR}}^i(X_{\mathbb{Q}_p}, \mathbb{Q}_p)$$

where now the filtration on the left is induced by the filtration

$$\text{Fil}^j \tilde{M}_i^+ = \{\mathbf{v} \in \tilde{M}_i^+ : \varphi(\mathbf{v}) \in q^j \tilde{M}_i^+\}$$

of  $\tilde{M}_i^+$  and the usual filtration on  $\mathbf{B}_{\text{dR}, \mathbb{Q}_p}^+$ . The identification of crystalline cohomology needs to be modified:

$$(6.9.3) \quad \tilde{M}_i^+ \otimes_{\tilde{\mathbf{A}}^+} W(\mathbb{F}_p)[p^{-1}] \cong H_{\text{crys}}^i(X_{\mathbb{F}_p}, \mathbb{Q}_p)$$

where the map  $\tilde{\mathbf{A}}^+ \rightarrow W(\mathbb{F}_p)$  is obtained by identify  $\tilde{\mathbf{A}}^+$  with  $W(\tilde{\mathbf{E}}^+)$  and then applying the map  $\tilde{\mathbf{E}}^+ \rightarrow \mathbb{F}_p$ . This has the effect of separating crystalline cohomology from de Rham cohomology.

**Remark 6.10.** The above discussion makes somewhat more liberal use than earlier parts of the paper of the fact that we work over  $\mathbb{Q}_p$ , rather than an arbitrary finite extension thereof. One cannot formally reduce to the case of  $\mathbb{Q}_p$  using Weil restriction because this does not preserve good reduction. The crystalline comparison theorem is now known over an arbitrary finite extension of  $\mathbb{Q}_p$  (we again refer to [4] for references) but we will not attempt to describe an analogue of Wach modules in that generality. (The Breuil-Kisin theory [12] can be used as an alternative.)

We now indicate how to incorporate the  $\mathbb{Q}$ -rational structure on de Rham cohomology into the theory of Wach modules, by analogy with the case of  $(\varphi, \Gamma)$ -modules.

**Definition 6.11.** Let  $S^+$  be the subring of  $\tilde{\mathbf{A}}^+$  consisting of those elements  $x$  for which for some nonnegative integer  $m$ , we have  $\theta_n(x) \in \mathbb{Q}(\mu_{p^{\min\{0, m-n\}}})$  for all  $n \in \mathbb{Z}$ . A *Wach module* over  $S^+$  is a finite  $p$ -free  $S^+$ -module  $M^+$  equipped with a semilinear  $\Gamma$ -action fixing  $M^+/\pi_0 M^+$  and a  $\varphi$ -semilinear  $\Gamma$ -equivariant map  $M^+ \rightarrow M^+[q^{-1}]$  which induces an isomorphism  $(\varphi^* M)[q^{-1}] \cong M^+[q^{-1}]$ .

We need the following analogue of Lemma 4.6.

**Lemma 6.12.** For  $n \in \mathbb{Z}$ , let  $R_n$  be the  $\ker(\theta_n)$ -adic completion of  $\mathbf{A}^+$ . Let  $M$  be a finite free module over  $\mathbf{A}^+$  equipped with the supremum norm (relative to  $|\bullet|_1$ ) for some basis. For  $n \in \mathbb{Z}$  and  $k \geq 0$ , equip  $M_{n,k} = M \otimes_R (R_n / \ker(\theta_n)^k)$  with the quotient topology. For  $n \in \mathbb{Z}$ , choose a quantity  $\epsilon_n > 0$  and a subset  $V_n$  of  $M_{n,\infty} = \varprojlim_{k \rightarrow \infty} M_{n,k}$  such that for each  $k \geq 0$  and each  $\mathbf{w}$  in the image of  $V_n \rightarrow M_{n,k}$ , the image of  $V_n \rightarrow M_{n,k+1}$  is dense in the inverse image of  $\mathbf{w}$  in  $M_{n,k+1}$ . Choose also a nonnegative integer  $m$ , a quantity  $\delta > 0$ , and an element  $\mathbf{v} \in M$ . Then there exists  $\mathbf{w} \in M$  such that  $\mathbf{v} - \mathbf{w} \in p^m M$ ,  $|\mathbf{v} - \mathbf{w}| < \delta$ , and for each integer  $n$ , the image of  $\mathbf{w}$  in  $M_{n,\infty}$  belongs to  $V_n$  and  $|\theta_n(\mathbf{v} - \mathbf{w})| < \epsilon_n$ .

*Proof.* We will construct elements  $\mathbf{w}_{-1}, \mathbf{w}_0, \mathbf{w}_1, \dots$  of  $M$  in such a way that  $\mathbf{v} - \mathbf{w}_n \in p^m M$ ,  $|\mathbf{v} - \mathbf{w}_n| < \delta$ , and for  $0 \leq n' \leq |n|$  the image of  $\mathbf{w}_n$  in  $M_{n',n-|n'|+1}$  belongs to the image of  $V_{n'} \rightarrow M_{n',n-|n'|+1}$  and  $|\theta_{n'}(\mathbf{v} - \mathbf{w}_n)| < \epsilon_{n'}$ . We will also ensure that  $|\mathbf{w}_n - \mathbf{w}_{n-1}| \leq p^{-n}$  for  $n \geq 0$ ; then the  $\mathbf{w}_n$  will converge to a limit  $\mathbf{w}$  having the desired property.

To begin, put  $\mathbf{w}_{-1} = \mathbf{v}$ . To continue, suppose we are given  $\mathbf{w}_{n-1}$  for some  $n \geq 0$ . Since  $\mathbf{A}^+[p^{-1}]$  is a principal ideal domain, we can find  $a_i, b_i \in S^+$  for  $i = -n, \dots, n$  and  $e \geq m$  for which

$$a_i \left( \prod_{|j| \leq n, j \neq i} \pi_j^{n-|j|+1} \right) + b_i \pi_i = p^e.$$

Using the density property of  $V_n$ , we may choose  $\mathbf{x}_i \in M$  so that

$$\mathbf{w}_n = \mathbf{w}_{n-1} + \sum_{i=-n}^n a_i \left( \prod_{j=-n}^n \pi_j^{n-|j|} \right) \left( \prod_{|j| \leq n, j \neq i} \pi_i \right) \mathbf{x}_i$$

has all of the desired properties: the image of  $\mathbf{w}_n$  in  $M_{n',n-|n'|+1}$  depends only on  $\mathbf{x}_{n'}$ , and the density property allows us to make  $\mathbf{x}_{n'}$  small enough to enforce the norm restrictions.  $\square$

**Theorem 6.13** (Crystalline comparison with rational structures). *Let  $X$  be a smooth proper  $\mathbb{Z}_{(p)}$ -scheme. Then for  $i \geq 0$ , there is a natural (in particular, functorial in  $X$ ) way to associate to  $X$  a Wach module  $M_i^+$  over  $S^+$  so as to obtain natural identifications*

$$(6.13.1) \quad D_{\text{et}}(M_i^+ \otimes_{S^+} \tilde{\mathbf{A}}) \cong H_{\text{et}}^i(X_{\overline{\mathbb{Q}_p}}, \mathbb{Z}_p)$$

of  $G_{\mathbb{Q}_p}$ -modules,

$$(6.13.2) \quad (M_i^+ \otimes_{S^+} B)^\Gamma \cong H_{\text{dR}}^i(X_{\mathbb{Q}}, \mathbb{Q})$$

of filtered  $\mathbb{Q}$ -vector spaces for the filtration induced on the left side by the filtration

$$\text{Fil}^j M_i^+ = \pi_0^j(\varphi^{-1})^* M_i^+$$

of  $M_i^+$  and the usual filtration on  $B^+$ , and

$$(6.13.3) \quad M_i^+ \otimes_{S^+} W(\mathbb{F}_p)[p^{-1}] \cong H_{\text{crys}}^i(X_{\mathbb{F}_p}, \mathbb{Q}_p)$$

of  $\varphi$ -modules over  $\mathbb{Q}_p$  (where  $S^+$  maps to  $W(\mathbb{F}_p)$  as in Remark 6.9).

*Proof.* This follows from Theorem 6.8 by imitating the proof of Theorem 4.10, using Lemma 6.12 in place of Lemma 4.6. We omit further details.  $\square$

## 7. REINTERPRETATION USING WITT VECTORS

Our principal motivation for formulating Theorem 5.5 is to make contact with an alternate point of view on the construction of  $p$ -adic period rings, which creates the possibility of more global constructions. For more discussion of this point of view, see [6].

**Definition 7.1.** For  $R$  a ring, let  $W_n(R)$  denote the ring of  $p$ -typical Witt vectors over  $R$  of length  $n$ . Let  $F : W_{n+1}(R) \rightarrow W_n(R)$  denote the Witt vector Frobenius map. Let  $\varprojlim (R)$  denote the inverse limit of the  $W_n(R)$  using the Frobenius maps as the transition maps.

**Remark 7.2.** The underlying set of  $W_n(R)$  is the set of  $n$ -tuples  $(x_1, \dots, x_{p^n-1})$  of elements of  $R$ . Its exotic ring structure is characterized by functoriality in  $R$  and the property that the *ghost map*  $w : W_n(R) \rightarrow R^n$  defined by the formula

$$(x_{p^i}) \mapsto (w_{p^i}), \quad w_{p^i} = \sum_{j=0}^i p^i x_{p^j}^{p^{j-i}}$$

is a ring homomorphism for the product ring structure on  $R^n$ . Similarly, the Frobenius map is characterized by naturality in  $R$  and the property that it corresponds to the left shift operator  $R^{n+1} \rightarrow R^n$  via the ghost map.

When  $p$  is invertible in  $R$ , the ghost map becomes an isomorphism. Consequently, in this case the kernel of  $F$  can be identified with  $R$  (as an additive group) by projection onto the first component. As a further consequence, the map taking each sequence in  $\varprojlim (R)$  to the corresponding sequence of first components is a bijection.

**Definition 7.3.** Let  $R$  be a ring equipped with a power-multiplicative nonarchimedean norm  $|\bullet|$ . Then the function  $|\bullet|_W$  on  $W_n(R)$  taking  $\underline{x} = (x_1, \dots, x_{p^n-1})$  to  $\max_i \{|x_i|^{p^{-i}}\}$  is again a power-multiplicative norm. For  $r > 0$ , let  $\varprojlim^{\dagger, r} (R)$  be the set of coherent sequences  $(\dots, \underline{x}_{1/p}, \underline{x}_1) \in \varprojlim (R)$  for which  $p^{-rn} |\underline{x}_{p^{-n}}|_W^{p^n} \rightarrow 0$  as  $n \rightarrow \infty$ .

**Theorem 7.4.** Let  $K$  be the  $p$ -adic completion of  $\mathbb{Q}_p(\mu_{p^\infty})$ . Define a map  $\tilde{\mathbf{A}}^{\dagger, 1} \rightarrow \varprojlim^{\dagger} (K)$  by assigning to  $x \in \tilde{\mathbf{A}}^{\dagger, 1}$  the element of  $\varprojlim^{\dagger} (K)$  whose sequence of first components is  $(\dots, \theta_{-1}(x), \theta_0(x))$  (see Remark 7.2). Then this map defines isomorphisms  $\tilde{\mathbf{A}}^{\dagger, 1} \cong \varprojlim^{\dagger, 1} (K)$  and  $\tilde{\mathbf{A}}^+ \cong \varprojlim^{\dagger} (\mathfrak{o}_K)$ .

*Proof.* See [6, Theorem 6.3]. □

**Remark 7.5.** The isomorphism  $\tilde{\mathbf{A}}^{\dagger, 1} \cong \varprojlim^{\dagger, 1} (K)$  induces isomorphisms

$$\begin{aligned} \tilde{S}_p &\cong \varprojlim^{\dagger, 1} (\mathbb{Q}_p(\mu_{p^\infty})) \\ S_p &\cong \varprojlim^{\dagger, 1} (\mathbb{Q}_p(\mu_{p^\infty})) \cap \bigcup_{m=0}^{\infty} \varprojlim_F W_n(\mathbb{Q}_p(\mu_{p^{m+n}})) \\ S' &\cong \varprojlim^{\dagger, 1} (\mathbb{Q}(\mu_{p^\infty})) \cap \bigcup_{m=0}^{\infty} \varprojlim_F W_n(\mathbb{Q}(\mu_{p^{m+n}})). \end{aligned}$$

It is also worth distinguishing between the ring  $\varprojlim (\mathbb{Z}_{(p)}[\mu_{p^\infty}])$ , which is the subring of  $\tilde{\mathbf{A}}^+$  consisting of those  $x$  for which  $\theta_n(x) \in \mathbb{Z}_{(p)}[\mu_{p^\infty}]$  for all  $n \leq 0$ , with the subring  $\varprojlim W(\mathbb{Z}_{(p)}[\mu_{p^\infty}])$ , which consists of those  $x$  for which  $\theta_n(x) \in \mathbb{Z}_{(p)}[\mu_{p^\infty}]$  for all  $n \in \mathbb{Z}$ .

**Remark 7.6.** By contrast, the ring  $S$  does not admit a natural interpretation in terms of Witt vectors. That is because the definition of  $S$  involves reductions modulo  $\pi_0^n$  for all  $n$ , whereas only the reduction modulo  $\pi_0$  appears in the context of ghost components. Consequently, Theorems 5.5 and 6.8 do not admit direct translations into the language of Witt vectors. It may be possible to achieve translations by working not merely with Witt vectors but with the *absolute de Rham-Witt complex* of Hesselholt [9].

One can also construct an analogue of  $\varprojlim^{\dagger,1}(\mathbb{Q}(\mu_{p^\infty}))$  of a more global nature, wherein the construction of [6] would be modified firstly by using big Witt vectors rather than  $p$ -typical Witt vectors, and secondly by keeping track of all of the places of  $\mathbb{Q}$  rather than only the  $p$ -adic place. It is tempting to imagine further that using the absolute de Rham-Witt complex, one can produce global analogues of analogues of  $(\varphi^{-1}, \Gamma_{\mathbb{Q}}, \mathbb{Z}_{(p)})$ -modules and inverse Wach modules which capture additional cohomological (or better,  $K$ -theoretic) structures. For instance, one would ideally like to capture the action of  $G_{\mathbb{Q}}$  on  $p$ -adic étale cohomology, not just the action of  $G_{\mathbb{Q}_p}$ . A first hint of this is given by [6, Theorem 6.4], which reformulates almost purity of ring extensions in terms of compatibility of formation of overconvergent Witt vectors with formation of finite étale extensions.

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